NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION (2010–2011)

MA1102R Calculus

April/May 2011 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [12 marks]

Find the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}$.

(b) $\displaystyle\lim_{x\to 0} \left(2e^{\frac{x}{x+1}}-1\right)^{\frac{x^2+1}{x}}$.

Question 2 [12 marks]

Evaluate the following integrals.

(a) $\displaystyle\int e^{\sqrt[3]{x}}\,dx$.

(b) $\displaystyle\int \frac{2x^3+5x^2+2x+2}{(x^2+2x+2)(x^2+2x-2)}\,dx$.

Question 3 [25 marks]

(a) Evaluate $\displaystyle\left.\frac{dy}{dx}\right|_{x=\pi/2}$ if $y = x(\sin x)^{\cos x}$.

(b) Prove that for all $x>0$,

$$e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

(c) (i) Express $\displaystyle\frac{d}{dx}(\tanh x)$ in terms of $\cosh x$.

(ii) Show that $\displaystyle\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$.

(d) Use an appropriate Riemann sum to evaluate the limit

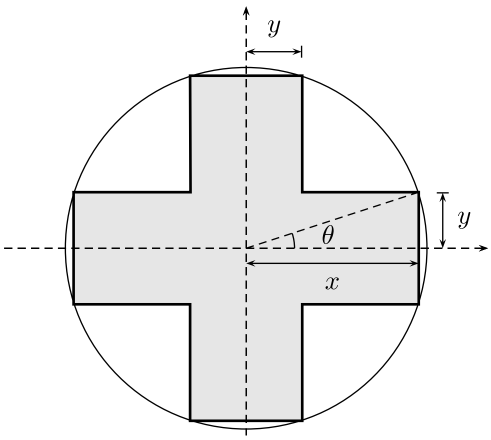
$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^5}{n^6} + \frac{\sqrt{i}}{n\sqrt{n}} \right).$$

Question 4 [10 marks]

Consider the symmetric cross inscribed in a circle of radius 1.

(i) Let θ be the angle as shown below. Show that the the area of the symmetric cross is given by $4\sin\theta(2\cos\theta - \sin\theta)$.

(ii) Find the maximum area of the symmetric cross. Correct your answer to 3 decimal places.

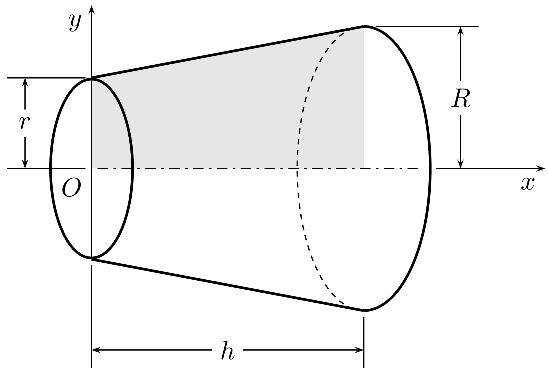


Question 5 [13 marks]

(a) Find the length of the curve $y^2 = x^3$ from the origin O to the point P where the line segment OP makes an angle of 45° with the x-axis.

(b) Show that the volume of the frustum of a right circular cone with height h, lower base radius R, and upper radius r is given by

$$V = \frac{\pi}{3}(r^2 + rR + R^2)h.$$



Question 6 [16 marks]

(a) Solve the following Bernoulli's equation with initial value condition:

$$xy^2 \frac{dy}{dx} + y^3 = \cos x \quad (x > 0), \qquad y(\pi) = 0.$$

(b) A cone-shaped water tank is shown below. When the tank is full, a valve is opened at the bottom of the tank. The depth of the water is halved after 1 hour.

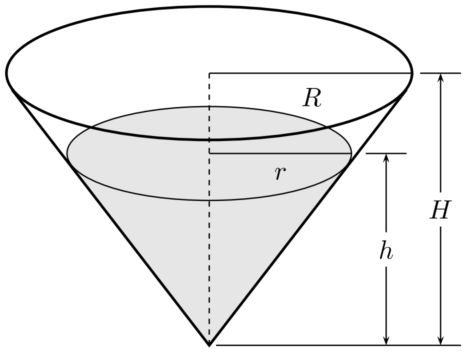
(i) Show that the depth of the water h and the time t can be modeled by

$$\frac{dh}{dt} = -\frac{c}{\sqrt{h^3}}$$

for some constant $c > 0$.

[*Torricelli's Law*: The rate at which water is flowing out is proportional to the square root of the water's depth.]

(ii) How long will it take for the tank to drain completely?



Question 7 [12 marks]

(a) Let f be a differentiable function defined on $\mathbb{R}$. Suppose that $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Prove that either $f'(x) < 0$ for all $x \in \mathbb{R}$, or $f'(x) > 0$ for all $x \in \mathbb{R}$.

(b) It is given that the following limit

$$\lim_{x \to \infty} \left(\int_0^{\pi/6} (\sin t)^x \, dt \right)^{1/x}$$

exists. Evaluate the limit.

End of Paper