Student Number:

A/U/T*								

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2010-2011

MA1101R LINEAR ALGEBRA I

April/May 2011 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper contains a total of **Five (5)** questions and comprises **Twenty Two (22)** printed pages.

3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.

4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
Total	

Question 1 (a) [10 marks]

Let $\boldsymbol{A}$ be a 3×5 matrix such that its row echelon form

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

(i) Write down a basis for the row space of $\boldsymbol{A}$.

(ii) Is it possible to find a basis for the column space of $\boldsymbol{A}$? Justify your answer.

(iii) Extend the basis found in (i) to a basis for $\mathbb{R}^5$. (Just write down the additional vectors.)

(iv) Find a basis for the nullspace of $\boldsymbol{A}$. Show your working.

(v) Given $\boldsymbol{v} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$ is a particular solution of the system $\boldsymbol{Ax} = \boldsymbol{b}$ for some $\boldsymbol{b} \in \mathbb{R}^3$. Write down the general solution of $\boldsymbol{Ax} = \boldsymbol{b}$.

Use the space below to write your answer and working

(More working spaces for Question 1 (a))

Question 1 (b) [5 marks]
Give examples of two matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ such that

they have the same first row $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, and same first column $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, but

$\text{rank}(\boldsymbol{B}) = 1$ and $\text{rank}(\boldsymbol{C}) = 2$.
Justify your answers.

Use the space below to write your answer and working

Question 1 (c) [5 marks]

Let $\boldsymbol{P}$ be an $m \times n$ matrix with rank $(\boldsymbol{P}) = n$, and $\boldsymbol{A}$ an $n \times p$ matrix. Suppose $m > n$. Show that $\text{rank}(\boldsymbol{PA}) = \text{rank}(\boldsymbol{A})$.

Use the space below to write your answer and working

Question 2 (a) [12 marks]

Let $\boldsymbol{A}$ be the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(i) Find all the eigenvalues of $\boldsymbol{A}$. Show your working.

(ii) Find a basis for the eigenspace of $\boldsymbol{A}$ associated with each of the eigenvalues. Show your working.

(iii) Write down an orthogonal matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^T$. Briefly explain how you obtain the answers.

Use the space below to write your answer and working

(More working spaces for Question 2 (a))

Question 2 (b) [4 marks]

Let $\boldsymbol{B} = \begin{pmatrix} a & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & a \end{pmatrix}$. Show that $\boldsymbol{B}$ is <u>not</u> diagonalizable for any values a and b.

Use the space below to write your answer and working

Question 2 (c) [4 marks]

Show that the only possible (real) eigenvalues of an orthogonal matrix are 1 and -1.

Use the space below to write your answer and working

Question 3 (a) [10 marks]

(i) Determine whether the subset $U = \{(1,0,1,0),(0,1,1,0),(0,0,0,1),(1,1,1,0)\}$ of $\mathbb{R}^4$ is linearly independent.

(ii) Express the subset $V = \{(a+b-2c, 0, c-a, b) \mid a,b,c \in \mathbb{R}\}$ of $\mathbb{R}^4$ in linear span form if possible (show your working). Is V a subspace of $\mathbb{R}^4$?

(iii) Show that the subset $W = \{(a,b,a+b,ab) \mid a,b \in \mathbb{R}\}$ of $\mathbb{R}^4$ is <u>not</u> a subspace of $\mathbb{R}^4$

Use the space below to write your answer and working

(More working spaces for Question 3 (a))

Question 3 (b) [6 marks]

Let $S = \{\boldsymbol{u}, \boldsymbol{v}\}$ and $T = \{\boldsymbol{u} - 2\boldsymbol{v}, 2\boldsymbol{u} + \boldsymbol{v}\}$ be two bases for a vector space V.

(i) Write down the transition matrix from T to S.

(ii) Write down the transition matrix from S to T.

(iii) Given the coordinate vector of $\boldsymbol{w} \in V$ with respect to T is $(\boldsymbol{w})_T = (1, 1)$. Find $(\boldsymbol{w})_S$.

Use the space below to write your answer and working

Question 3 (c) [4 marks]

Let $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be subspaces of $\mathbb{R}^5$ such that $\dim(U \cap V) = 2$. Suppose W is the smallest subspace of $\mathbb{R}^5$ that contains both U and V. Determine all possible dimensions of W. Justify your answers.

Use the space below to write your answer and working

Question 4 (a) [12 marks]

Let V be a subspace of $\mathbb{R}^3$ with a basis $\{(1,1,0),(1,3,2)\}$.

(i) What geometrical object does V represent in the xyz-space?

(ii) Applying Gram-Schmidt process to $\{(1,1,0),(1,3,2)\}$ to find an orthogonal basis for V. Show your working.

(iii) Let $\boldsymbol{w} = (1,1,1)$. Find $\boldsymbol{p}$, the projection of $\boldsymbol{w}$ onto V. Show your working.

(iv) Find a vector $\boldsymbol{x}$ different from $\boldsymbol{w}$ in (iii), and not belonging to V, such that $d(\boldsymbol{w},\boldsymbol{p}) = d(\boldsymbol{x},\boldsymbol{p})$. Justify your answer.

Use the space below to write your answer and working

(More working spaces for Question 4 (a))

Question 4 (b) [4 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find a least squares solution of $\boldsymbol{Ax} = \boldsymbol{b}$. Show your working.

Use the space below to write your answer and working

Question 4 (c) [4 marks]

True or false: Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Ax} = \boldsymbol{c}$ are both inconsistent systems that have exactly the same least squares solutions. Then $\boldsymbol{b} = \boldsymbol{c}$.
Justify your answer.

Use the space below to write your answer and working

Question 5 (a) [15 marks]

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x \\ y \end{pmatrix}.$$

(i) Write down the standard matrix for T.

(ii) Write down the range $R(T)$ of T as a linear span.

(iii) Find the kernel of T. Justify your answer.

(iv) Suppose $S : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with standard matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Write down the formula for the composition $T \circ S$.

(v) Is it possible to find a linear transformation $T' : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T' \circ T$ is the identity map on $\mathbb{R}^2$? Justify your answer.

(vi) Is it possible to find a linear transformation $Q : \mathbb{R}^3 \to \mathbb{R}^2$ such that $R(T \circ Q) = \text{span}\{(1, 1, 1)^T\}$? Justify your answer.

Use the space below to write your answer and working

(More working spaces for Question 5 (a))

Question 5 (b) [5 marks]
Given $P : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. Let P^m denote the composition of P with itself m times, i.e.

$$P^m = \underbrace{P \circ P \circ \cdots \circ P}_{m \text{ times}}$$

Show that there is an integer $k > 0$ such that

$$R(P^k) = R(P^{k+1}) = R(P^{k+2}) = \cdots .$$

Use the space below to write your answer and working

(More working spaces. Please indicate the question numbers clearly.)

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