MA4262 Mid-term test 2010

(1) Suppose f is an extended real valued function on a Borel measurable set E such that

$$\{x \in E : f(x) > \frac{k}{5^n}\} \text{ is a Borel set for all } k \in \mathbb{Z}, n \in \mathbb{N}.$$

Show that f is Borel measurable. [15]

(2) If f is an integrable function on a measurable set E and $E = \cup_{i=1}^{\infty} E_i$ such that each E_i is measurable and $E_i \cap E_j = \emptyset$ when $i \neq j$, show that $\sum_{i=1}^{\infty} \int_{E_i} f = \int_E f$. [15]

(3) If f is a measurable function on a set E with $m(E) < \infty$ and $|f(x)| < \infty$ for almost all $x \in E$, show that given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and a compact set K such that $m(E \setminus K) < \varepsilon$ and $|f(x)| < N$ for all $x \in K$. [15].

(4) For $a, b \in [0, 1)$, we define $a \dot{+} b = a + b$ if $a + b < 1$ and $a + b - 1$ if $a + b \geq 1$. If $A \subset [0, 1)$ and $A \dot{+} a = \{x \dot{+} a : x \in A\}$, show that $m^*(A \dot{+} a) = m^*(A)$ (for any $a \in [0, 1)$). [15].

(5) Prove or disprove each of the following statements. (Answer not more than five of them) [40]

(a) If f is absolutely continuous function on $[a, b]$, then it is a difference of two continuous nondecreasing function on $[a, b]$.

(b) If f is a function of bounded variation on $[a, b]$, then f is continuous on $[a, b]$

(c) If f is integrable on E, then given any $\varepsilon > 0$, there exists a compact set $K \subset E$ such that $\int_{E \setminus K} |f| < \varepsilon$.

(d) If $f = g$ a.e. and g is continuous on $[0, 1]$, then f is continuous a.e..

(e) If E is measurable, then given any $\varepsilon > 0$, there exists a finite union of pairwise disjoint closed intervals G such that $m(E \triangle G) < \varepsilon$.

(f) A step function on $[0, 1]$ is always measurable.

(g) If E is a measurable set in $\mathbb{R}$, then $m(E) = \sup\{m(K) : K \text{ is a compact subset of } E\}$.

(6) If $f_k \geq 0$ for all k and $f_k \to f$ in measure, show that [Bonus 5]

$$\liminf_{k \to \infty} \int f_k \geq \int f.$$