NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2010-2011

MA2213 Numerical Analysis I

November 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring one piece of A4-sized two-sided help sheet into the examination room.

2. This examination paper contains a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

3. Answer **ALL** questions in **Section A**. Each question in Section A carries 20 marks.

4. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1. [20 marks]

(a) Solve the following system using Gaussian elimination with partial pivoting and four digits rounding arithmetic.

$$\begin{aligned} 0.729x + 0.81y + 0.9z &= 0.6867 \\ x + y + z &= 0.8338 \\ 1.331x + 1.21y + 1.1z &= 1. \end{aligned}$$

(b) Let

$$A = \begin{bmatrix} 2.25 & -3 & 4.5 \\ -3 & 5 & -10 \\ 4.5 & -10 & 34 \end{bmatrix}.$$

Find a factorization of the form $A = LL^T$ where L is lower triangular with positive diagonal entries. Explain why if there is no such a factorization.

Question 2. [20 marks]

(a) Find the Hermite interpolating polynomial for the following data using Newton's divided-difference formula.

x	0	1	3
$f(x)$	0	1	-1
$f'(x)$	1	0	0

(b) Let $i_0, i_1, \cdots, i_n$ be a rearrangement of the integers $0, 1, \cdots, n$. Suppose $x_0, x_1, \cdots, x_n$ are distinct points. Show that

$$f[x_{i_0}, x_{i_1}, \cdots, x_{i_n}] = f[x_0, x_1, \cdots, x_n],$$

where $f[x_{i_0}, x_{i_1}, \cdots, x_{i_n}]$ and $f[x_0, x_1, \cdots, x_n]$ are nth divided differences.

Question 3. [20 marks]

(a) Determine constants a, b, c that will produce a quadrature formula

$$\int_{-1}^{1} xf(x)dx = af(-c) + bf(c)$$

that has the highest degree of precision. Determine the degree of precision.

(b) Determine the four missing entries in the following table for the approximation of the integral

$$\int_{0}^{1} \frac{4}{1+x^2}dx$$

with Romberg integration. (Keep 6 decimal digits.)

n	h	$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$
1	1	3.000000			
2	1/2	$R_{2,1}$	$R_{2,2}$		
3	1/4	$R_{3,1}$	3.141568	3.142117	
4	1/8	3.138988	3.141592	3.141594	$R_{4,4}$

SECTION B

Answer not more than **two** *questions from this section. Section B carries a total of 40 marks.*

Question 4. [20 marks]

(a) Determine the discrete least squares trigonometric polynomial approximation $S_6(x)$ in the form

$$S_6(x) = \frac{a_0}{2} + a_6 \cos 6x + \sum_{k=1}^{5}(a_k \cos kx + b_k \sin kx),$$

for $f(x) = \cos 7x + \sin 7x$ on the interval $[-\pi, \pi]$ with 16 points (x_j, y_j), where $x_j = -\pi + \frac{j}{8}\pi$ for $j = 0, 1, \cdots, 15$.

(b) Let $\{(x_j, y_j)\}_{j=0}^{2m-1}$ be $2m$ points where $x_j = -\pi + \frac{j}{m}\pi, y_j = f(x_j)$ for $j = 0, 1, \cdots, 2m-1$. Suppose an interpolatory **trigonometric polynomial** $P(x)$ of degree m and a least squares **trigonometric polynomial** $Q(x)$ of degree $n < m$ are obtained to approximate $f(x)$ with the given $2m$ points .
The integral $\int_{-\pi}^{\pi} f(x)dx$ can then be approximated by integrating $P(x)$ or $Q(x)$. Which do you think will give a better approximation, $\int_{-\pi}^{\pi} P(x)dx$ or $\int_{-\pi}^{\pi} Q(x)dx$? What happens when $m \to \infty$? Justify your answer.

Question 5. [20 marks]

Let $x_0, x_1, \cdots, x_n$ be distinct real points, and consider the following interpolation problem. Choose a function

$$P(x) = \sum_{k=0}^{n} c_k e^{kx}$$

such that $P(x_i) = y_i$, for $i = 0, 1, \cdots, n$ with the given data $\{y_i\}$.

Show that this problem can be reduced to an ordinary polynomial interpolation and then use Lagrange interpolation polynomial to find a general formula for $P(x)$.

Question 6. [20 marks]

The set of Legendre polynomials $\{P_0(x), P_1(x), \cdots, P_n(x), \cdots,\}$ is orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = 1$ where $P_k(x)$ is a polynomial of degree k. That is, for any $i \neq j$, one has

$$\int_{-1}^{1} P_i(x)P_j(x)dx = 0.$$

Moreover, any Legendre polynomial $P_k(x)$ has k distinct roots, lying in the interval $(-1, 1)$. Let $x_1, x_2, \cdots, x_n$ be the roots of the nth Legendre polynomial $P_n(x)$. Define the following quadrature formula to approximate integral $\int_{-1}^{1} f(x)dx$,

$$G(f) = \sum_{i=1}^{n} c_i f(x_i)$$

where

$$c_i = \int_{-1}^{1} \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} dx \quad \text{for } i = 1, 2, \cdots, n.$$

(i) Show that the quadrature formula is exact for any polynomial $h(x)$ of degree less than n.

(ii) Show further that the quadrature formula is exact for any polynomial $h(x)$ of degree at least n but less than $2n$ using the fact that $h(x)$ can be written as $h(x) = q(x)P_n(x) + r(x)$ where $P_n(x)$ is the nth Legendre polynomial and $q(x), r(x)$ are polynomials of degree less than n.

(iii) Find a quadrature formula with degree of precision 5 given the Legendre polynomial

$$P_3(x) = x^3 - \frac{3}{5}x.$$

END OF PAPER