NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2010-2011

MA2108 Mathematical Analysis I

November 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring one piece of A4-sized two-sided help sheet into the examination room.

2. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **SIX (6)** printed pages.

3. Answer **ALL** questions in **Section A**. Section A carries a total of 70 marks.

4. Answer not more than **TWO (2)** questions from **Section B**. Section B carries a total of 30 marks.

5. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 70 marks.*

Question 1.

(a) The sequence (a_n) is defined by

$$a_1 = 2, \quad a_{n+1} = \frac{4}{5}(a_n - 1) \quad \text{for all } n \in \mathbb{N}.$$

Prove that (a_n) converges and find its limit. [7 marks]

(b) Let

$$x_n = \frac{(2n^3 + 3)\cos\left(\frac{n\pi}{3}\right)}{n(6n^2 + 5)}, \quad \text{for all } n \in \mathbb{N}.$$

(i) Find $\limsup x_n$ and $\liminf x_n$. [6 marks]

(ii) Is (x_n) convergent? Justify your answer. [2 marks]

Question 2.

(a) Determine whether the following series converge or diverge. Justify your answers.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n(3n^3 + 5)}{4n^5\sqrt{n} - 3n^2 + 2}$. [4 marks]

(ii) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{2n}{1 + 2n}\right)^{n^2}$. [4 marks]

(b) Prove that if the series $\sum_{n=1}^{\infty} x_n^2$ converges, then the series $\sum_{n=1}^{\infty} \frac{x_n}{n}$ converges absolutely.

[4 marks]

(c) Suppose that the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(i) Prove that if $a_n \geq 0$ for all $n \in \mathbb{N}$ and (a_{n_k}) is a subsequence of (a_n), then the series $\sum_{k=1}^{\infty} a_{n_k}$ is also convergent.

[6 marks]

(ii) Without the assumption that $a_n \geq 0$ for all $n \in \mathbb{N}$, does the series $\sum_{k=1}^{\infty} a_{n_k}$ necessarily converge? Justify your answer.

[4 marks]

Question 3.

(a) Use the definition of limit to prove the following:

(i) $\lim_{x \to 2} \frac{1}{3x-4} = \frac{1}{2}$. [6 marks]

(ii) $\lim_{x \to 4^+} \frac{x+1}{x-4} = \infty$. [4 marks]

(b) In each part, either evaluate the limit or show that the limit does not exist.

(i) $\lim_{x \to 1} \cos\left(\frac{1}{\sqrt{x}-1}\right)$. [4 marks]

(ii) $\lim_{x \to 1^+} \frac{[7-5x]}{1+x^2}$. [4 marks]

Here $[7-5x]$ denotes the greatest integer less than or equal to $7-5x$.

Question 4.

(a) Let I be an interval. The function $f : I \to \mathbb{R}$ has the following property: There exists $C > 0$ such that

$$|f(x) - f(y)| < C|x - y| \quad \text{for all } x, y \in I.$$

Prove that f is uniformly continuous on I. [5 marks]

(b) In each part, determine whether the function is uniformly continuous on $(0, 1)$. Justify your answers.

(i) $g(x) = \cos\left(\frac{1}{x}\right)$. [5 marks]

(ii) $h(x) = x \sin\left(\frac{1}{x}\right)$. [5 marks]

SECTION B

Answer not more than **two** *questions from this section. Section B carries a total of 30 marks.*

Question 5.

(a) Let (a_n) be a sequence of positive numbers. Prove that if the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ is bounded and

$$\limsup \frac{a_{n+1}}{a_n} < 1,$$

then the series $\sum_{n=1}^{\infty} a_n$ converges. [7 marks]

(b) Suppose that the sequence (x_n) has the following properties:

(i) $x_n > 0$ and $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.

(ii) $\lim\limits_{n\to\infty} x_n = 0$.

Prove that the series

$$\sum_{n=1}^{\infty}(-1)^{[(n+1)/2]}\, x_n = -x_1 - x_2 + x_3 + x_4 - x_5 - x_6 + x_7 + x_8 - \cdots$$

converges. Here $[(n+1)/2]$ is the greatest integer less than or equal to $(n+1)/2$.

[8 marks]

Question 6.

(a) Suppose that the function $h : (0,\infty) \to \mathbb{R}$ has the following properties:

(i) h is continuous at 1.

(ii) $h(x) = h(x^3)$ for all $x > 0$.

Prove that h is a constant function on $(0,\infty)$. [7 marks]

(b) Let the function $f : (0,1) \to \mathbb{R}$ be bounded on $(0,1)$. For each $x \in (0,1)$, let

$$g(x) = \sup\{f(t) : t \in (0,x)\}.$$

(i) Explain why the limit $L = \lim_{x\to 0^+} g(x)$ exists. [2 marks]

(ii) Prove that for every $\varepsilon > 0$, there exists $0 < \delta < 1$ such that

$$f(x) < L + \varepsilon$$

for all $x \in (0,\delta)$. [3 marks]

(iii) Prove that for every $\varepsilon > 0$ and for every $0 < \delta_1 < 1$, there exists $x_1 \in (0,\delta_1)$ such that

$$f(x_1) > L - \varepsilon.$$

[3 marks]

Question 7.

(a) The function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Prove that there exists $c \in \mathbb{R}$ such that

$$f(c) = \frac{1}{3}f(x_1) + \frac{1}{12}f(x_2) + \frac{5}{12}f(x_3) + \frac{1}{6}f(x_4).$$

[7 marks]

(b) The function $g : [0, \infty) \to \mathbb{R}$ is uniformly continuous on $[0, \infty)$ and for any $x \geq 0$,

$$\lim_{n\to\infty} g(x + n) = 0.$$

Prove that

$$\lim_{x\to\infty} g(x) = 0.$$

[8 marks]

END OF PAPER