NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2010-2011

MA2101 Linear Algebra II

November 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **SIX (6)** printed pages.
2. Answer **ALL** questions.
3. Marks for each question are indicated at the beginning of the question. The marks for questions are not necessarily the same.
4. Calculators may be used. However, various steps in the calculations should be systematically laid out.

Question 1 [12 marks]

Let $f : V \to W$ be a linear transformation.

(a) Given a vector $\mathbf{w}_1 \in W$, we define its f-preimage as

$$f^{-1}(\mathbf{w}_1) := \{\mathbf{v} \in V \mid f(\mathbf{v}) = \mathbf{w}_1\}.$$

Show that for every $\mathbf{v}_1 \in V$ we have

$$f^{-1}(f(\mathbf{v}_1)) = \mathbf{v}_1 + \mathrm{Ker}(f).$$

(b) Does the following equality of cardinalities

$$|f^{-1}(\mathbf{w}_1)| = |f^{-1}(\mathbf{w}_2)|$$

hold for *all* $\mathbf{w}_i \in W$? Justify your answer.

(c) If U is a vector subspace of W, then its f-preimage

$$f^{-1}(U) := \{\mathbf{v} \in V \mid f(\mathbf{v}) \in U\}$$

is known to be a vector subspace of V. Show that

$$\dim f^{-1}(U) \leq \dim U + \dim \mathrm{Ker}(f).$$

Question 2 [12 marks]

Let

$$V = P_3[x] = \{\sum_{i=0}^{2} a_i x^i \mid a_i \in \mathbf{R}\}$$

be the vector space of real polynomials of degree ≤ 2. Let $B = (1, x, x^2)$ be the standard basis of V.

(i) Find a linear transformation

$$T : V \to V$$

such that the representation matrix $[T]_B$ satisfies

$$[T]_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(ii) Calculate

$$T(1), \quad T(x), \quad T(x^2).$$

(iii) Find a general formula for

$$T(a_0 + a_1 x + a_2 x^2).$$

(iv) Is the T in (i) unique? Justify your answer.

Question 3 [12 marks]

Let $A \in M_6(\mathbf{R})$ be a real matrix. Suppose that its characteristic and minimal polynomials are given as follows:

$$p_A(x) = (x^2 + 1)(x - 2)^4, \quad m_A(x) = (x^2 + 1)(x - 2)^2.$$

(i) Does A have a Jordan canonical form J in the real matrix space $M_6(\mathbf{R})$? If the answer is yes, find *all* such forms J. Justify your answer.

(ii) Does A have a Jordan canonical form J in the complex matrix space $M_6(\mathbf{C})$? If the answer is yes, find *all* such forms J. Justify your answer.

Question 4 [12 marks]

Consider the polynomial

$$f(x) = (x-2)(x-3)(x-4).$$

Let $C \in M_4(\mathbf{R})$ be a real matrix such that $f(C) = 0$.

(i) Show that there is an invertible matrix $P \in M_4(\mathbf{R})$ such that $P^{-1}CP$ equals a diagonal matrix J.

(ii) Find *all* possible minimal polynomials $m_C(x)$ of C. Justify your answers.

(iii) Find *all* possible characteristic polynomials $p_C(x)$ of C. Justify your answers.

Question 5 [12 marks]

Consider the symmetric matrix

$$D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(i) Determine the characteristic polynomial $p_D(x)$ of D.

(ii) Find all eigenvalues of D.

(iii) Calculate all eigenspaces of D.

(iv) Find an orthogonal matrix Q such that $Q^{-1}DQ = Q^tDQ$ equals a diagonal matrix J.

(v) Determine the above J.

(vi) Determine the minimal polynomial $m_D(x)$ of D.

Question 6 [12 marks]

Note. For the differential equation

$$z'(x) + p(x)\, z = q(x)$$

you may assume, without proof, that its general solution is given as

$$z(x) = \frac{1}{\mu}\Big(\int \mu\, q(x)\, dx \,+\, C \Big)$$

with

$$\mu := e^{\int p(x)\, dx}.$$

Now consider matrices

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let $A \in M_3(\mathbf{R})$ be a real matrix such that $P^{-1}AP = J$. Let $y_i = y_i(x)$ $(i = 1, 2, 3)$ be differentiable functions. Find all solutions of the differential equation

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Question 7 [12 marks]

Let $T_i : V \to V$ $(i = 1, 2, 3)$ be linear operators on an n-dimensional vector space V defined over the real number field $\mathbf{R}$.

(a) Show that if T_1 is self-adjoint (i.e., $T_1^* = T_1$), then every eigenvalue of T_1 (i.e., zero of the characteristic polynomial $p_{T_1}(x)$ of T_1) is a real number.

(b) Show that if T_2 is orthogonal (i.e., $T_2^*\, T_2 = I_V$), then every eigenvalue of T_2 (i.e., zero of the characteristic polynomial $p_{T_2}(x)$ of T_2) has modulus equal to 1.

(c) Show that if T_3 is both self-adjoint and orthogonal, then $T_3 \circ T_3 = I_V$.

Question 8 [16 marks]

Consider the following statements. If a statement is true, give a *simple* proof (you may use results in lecture notes or tutorials). If a statement is false, disprove it *conceptually* or by a *concrete* counter-example.

(i) Let T_1, T_2 be two distinct linear operators on a vector space V with representation matrices $A_1 = [T_1]_{B_1}$ and $A_2 = [T_2]_{B_2}$ relative to two distinct bases B_1, B_2 of V. If $T_1 \circ T_2 = T_2 \circ T_1$ then $A_1 A_2 = A_2 A_1$.

(ii) Every unitary matrix $P \in M_n(\mathbf{C})$ is diagonalizable.

(iii) If T_3, T_4 are commutative linear operators on a vector space V and J_i $(i = 3, 4)$ is a Jordan canonical form of T_i, then there is a common basis B of V such that the representation matrices satisfy: $[T_3]_B = J_3$, $[T_4]_B = J_4$.

(iv) For every invertible complex matrix $D \in M_n(\mathbf{C})$, we can write $D = SU = US$ where $S \in M_n(\mathbf{C})$ is diagonalizable and $U \in M_n(\mathbf{C})$ has 1 as its only eigenvalue.

(v) If every entry c_{ij} of a matrix $C = (c_{ij}) \in M_n(\mathbf{R})$ is a positive real number, then C is positive definite.

(vi) For every complex matrix $E \in M_n(\mathbf{C})$, there is a unitary matrix U such that U^*EU is lower triangular.

(vii) If two real matrices $F_i \in M_3(\mathbf{R})$ have the same minimal polynomial $m(x) = (x-1)(x-2)$, then the Jordan canonical forms of F_1 and F_2 are the same after re-ordering the Jordan blocks.

(viii) If two complex matrices $A_i \in M_4(\mathbf{C})$ $(i = 3, 4)$ have the same characteristic polynomial $p(x) = (x^2 + 1)(x^2 + 2)$, then A_3 and A_4 are similar.

END OF PAPER