NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER I EXAMINATION 2010-2011

MA1104 Multivariable Calculus

November 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring two pieces of A4-size help sheets into the examination room.

2. This examination paper contains a total of **TEN (10)** questions and comprises **SIX (6)** printed pages.

3. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1. [10 marks]

(i) Find the equation of the plane W which contains the points $(3, -1, 2), (8, 2, 4)$ and $(-1, -2, -3)$.

(ii) Find all the planes U which are parallel to the plane W in Part (i) and whose distance from U to W is $\sqrt{3}$.
Write down the equation of each U.

(An equation of the plane should be expressed in the form $ax + by + cz = d$.)

Question 2. [10 marks]

Let C denote the curve in $\mathbb{R}^3$ which is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y = z$.

(i) Describe C in the form $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $0 \leq t \leq 2\pi$.

(ii) Find the parametric equations of the tangent line of C at $(1, 0, 0)$.

(iii) Compute the curvature of C at $(1, 0, 0)$.

Question 3. [12 marks]

Consider the following limits:

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + 4y^4}$.

(ii) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2 z}{x^2 + 4y^4 + 9z^6}$.

In each part, determine whether the limit exists.
If it exists, compute its value.
If it does not exist, give a proof.

Question 4. [5 marks]

(i) Find the tangent plane of the surface

$$x^2 - 2y^2 + z^2 + yz = 2$$

at the point $P(2, 1, -1)$.

(ii) Find parametric equations of the line which is perpendicular to the tangent plane in (i) and passes through $P(2, 1, -1)$.

Question 5. [10 marks]

A supplier would like to make a rectangular box without top cover of length x cm, breadth y cm and height z cm out of 300 cm^2 of cardboard.

Determine the box with the largest volume that he could make.

Give the dimensions of x, y, z and the volume of this box.

Question 6. [8 marks]

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Suppose $f(x, y)$ is differentiable at (a, b).
Consider the limit

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - f(a, b) - \nabla f(a, b) \cdot \langle x - a, y - b\rangle}{\sqrt{(x - a)^2 + (y - b)^2}}.$$

If the limit exists, compute its value.
If the limit does not exist, give a counterexample.

Question 7. [10 marks]

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that

$$f(tx, ty) = t^7 f(x, y)$$

for all $x, y, t \in \mathbb{R}$.

(i) Show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = Cf(x, y)$$

for some constant C.
Determine the value of C.

(ii) Show that

$$f_x(tx, ty) = t^n f_x(x, y)$$

for some integer n.
Determine the value of n.

Hint: For (i), consider $\frac{d}{dt} f(at, bt)$.

Question 8. [15 marks]

(i) Let $R = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ be the unit square on the xy-plane. Determine the domain G in the uv-plane so that $T : G \to R$ given by

$$T(u,v) = \left(\frac{\sin u}{\cos v}, \frac{\sin v}{\cos u}\right)$$

is a bijection.
Sketch the domain G.

(ii) Show that $T : G \to R$ is a injection.

(iii) Compute the integral

$$\iint_R \frac{1}{1-(xy)^2} dxdy$$

by making the substitution $x = \dfrac{\sin u}{\cos v}$ and $y = \dfrac{\sin v}{\cos u}$.

(iv) Using Part (ii) or otherwise, find the sum

$$S = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hint: You may assume that

$$u = \sin^{-1}\sqrt{\frac{1-y^2}{x^{-2}-y^2}}, \quad v = \sin^{-1}\sqrt{\frac{1-x^2}{y^{-2}-x^2}}, \quad \sin X = \cos(\frac{\pi}{2} - X).$$

Question 9. [10 marks]

Let k be a real number and let

$$\mathbf{F}(x,y,z) = \langle e^{xy}(xy+k)\sin z,\ x^2e^{xy}\sin z,\ xe^{xy}(\cos z)+2z\rangle$$

be a vector conservative field in $\mathbb{R}^3$.

(i) Determine the constant k.

(ii) Find a potential function $f(x,y,z)$ of $\mathbf{F}(x,y,z)$, ie. $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$.

(iii) Let C be the arc $\mathbf{r}(t) = \langle \sin t, \cos^2 t, t\rangle$ for $0 \le t \le \frac{\pi}{2}$.
Compute the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Question 10. [10 marks]

Let

$$\mathbf{F}(x,y,z) = \left\langle z^2x, \frac{1}{3}y^3 + \sin z, x^2z + y^2 \right\rangle.$$

Evaluate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the unit hemisphere above the xy-plane.

END OF PAPER