Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2010–2011

MA1101R LINEAR ALGEBRA I

November 2010 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **SIX (6)** questions and comprises **TWENTY-THREE (23)** printed pages.
3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.
4. Total marks for this examination is **100**. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	Total
Marks							

Question 1 [15 marks]

(a) Let $S_1 = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ be a set of vectors in $\mathbb{R}^n, n > 3$. If $S_2 = \{\boldsymbol{u} - 2\boldsymbol{v}, \boldsymbol{v} - 2\boldsymbol{w}, \boldsymbol{w}\}$, show that

$$\text{span}(S_1) = \text{span}(S_2).$$

(b) Let $T_1 = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ be a set of *linearly independent* vectors in $\mathbb{R}^3$ and $T_2 = \{\boldsymbol{u}+2\boldsymbol{v}, \boldsymbol{v}+2\boldsymbol{w}, \boldsymbol{u}+\boldsymbol{w}\}$.

(i) Determine whether T_2 is also linearly independent. Justify your answer.

(ii) Is it true that $\text{span}(T_1) = \text{span}(T_2)$? Justify your answer.

(c) Let $X = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of linearly independent vectors in $\mathbb{R}^n, n > k$. Suppose $\boldsymbol{v}$ is a vector in $\mathbb{R}^n$ and $\boldsymbol{v} \notin \text{span}(X)$, show that the set $X \cup \{\boldsymbol{v}\}$ is linearly independent.

Question 2 [15 marks]

For a general matrix $\boldsymbol{A}$, define the *left nullspace* of $\boldsymbol{A}$ as the solution set of the linear system $\boldsymbol{x}\boldsymbol{A} = \mathbf{0}$.

(a) Let $\boldsymbol{B} = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 3 & 2 \end{pmatrix}$.

(i) Find a basis for the row space of $\boldsymbol{B}$.

(ii) Find a basis for the column space of $\boldsymbol{B}$.

(iii) Find a basis for the nullspace of $\boldsymbol{B}$.

(iv) Find a basis for the *left nullspace* of $\boldsymbol{B}$.

(b) For a general matrix $\boldsymbol{A}$, prove that if the *left nullspace* of $\boldsymbol{A}$ has only the trivial solution, then the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent for every vector $\boldsymbol{b}$.

Question 3 [15 marks]

(a) Let $n \geq 2$ be a positive integer and a be a constant. Define a square matrix $\boldsymbol{F}_n = (f_{ij})_{n\times n}$ by

$$f_{ij} = \begin{cases} 1+a & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

(i) Write down explicitly $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ and compute $\det(\boldsymbol{F}_2)$ and $\det(\boldsymbol{F}_3)$.

(ii) What is the value of $\det(\boldsymbol{F}_n)$ for general $n \geq 2$? Justify your answer.

(b) Determine if the following statement is true. Justify your answer.

If $\boldsymbol{A}$ is an $n \times n$ matrix satisfying $\boldsymbol{A}^T = -\boldsymbol{A}$, then $\boldsymbol{A}$ is not invertible.

Question 4 [15 marks]

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}3\\2\\2\end{pmatrix}, \quad T\left(\begin{pmatrix}2\\1\\1\end{pmatrix}\right) = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

(i) Write down the standard matrix for T.

(ii) Find the kernel of T.

(iii) Find the kernel of $T \circ T$.

(iv) Find a basis for the subspace V given by

$$V = R(T) \cap \mathrm{Ker}(T).$$

($R(T)$ denotes the range of T and $\mathrm{Ker}(T)$ denotes the kernel of T.)

(b) Let $\boldsymbol{u}$ be any unit vector in $\mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$S(\boldsymbol{x}) = 2(\boldsymbol{u} \cdot \boldsymbol{x})\boldsymbol{u} - \boldsymbol{x}.$$

Prove that $S \circ S$ is the identity transformation.

Question 5 [20 marks]

Let V be a vector space with basis $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $\boldsymbol{w} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$.

(i) Show that $\boldsymbol{w} \notin V$.

(ii) Use the Gram-Schmidt process to find an orthogonal basis T for the V.

(iii) Find the transition matrix from the basis S to the basis T.

(iv) Find the transition matrix from the basis T to the basis S.

(v) Let $\boldsymbol{p}$ be the projection of $\boldsymbol{w}$ onto the space spanned by T. Find $\boldsymbol{p}$ and $[\boldsymbol{p}]_T$.

(vi) Hence or otherwise find $[\boldsymbol{p}]_S$.

Question 6 [20 marks]

(a) Let $\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$. Find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{C} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$. (You do not need to find $\boldsymbol{P}^{-1}$.)

(b) Let $\boldsymbol{A}$ be a 3×3 matrix with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and corresponding eigenspaces $E_{\lambda_1}, E_{\lambda_2}, E_{\lambda_3}$.

(i) Prove that $E_{\lambda_1} \cap E_{\lambda_2} = \{\boldsymbol{0}\}$.

(ii) Is $E_{\lambda_1} \cup E_{\lambda_2} \cup E_{\lambda_3} = \mathbb{R}^3$? Justify your answer.

(c) An $n \times n$ symmetric matrix $\boldsymbol{S}$ is said to be *positive definite* if for every non-zero vector $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\boldsymbol{x}^T\boldsymbol{S}\boldsymbol{x} > 0.$$

Prove that a symmetric matrix $\boldsymbol{M}$ is *positive definite* if and only if all the eigenvalues of $\boldsymbol{M}$ are strictly positive. (You may assume that any symmetric matrix is orthogonally diagonalizable.)

(d) Prove that any $n \times n$ matrix $\boldsymbol{Q}$ is invertible if and only if $\boldsymbol{Q}^T\boldsymbol{Q}$ is *positive definite.*

End of Paper