Ph.D. Qualifying Examination 2011 January (Analysis)

(1) If $\{\phi_k : k \in \mathbb{N}\}$ is an orthonormal family of functions in a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, show the Bessel's inequality: [8 marks]

$$\sum_{k=1}^{\infty} |\langle x, \phi_k \rangle|^2 \leq \langle x, x \rangle \quad \text{for all } x \in H.$$

(2) Let $\{x_n\}$ be a bounded sequence of real numbers and let S be the collection of limit points of convergent subsequences of $\{x_n\}$. Show that S is closed. [10 marks]

(3) Explain why there is no differentiable function f on $\mathbb{R}$ such that $f' = \chi_{\mathbb{Q}}$ on $\mathbb{R}$. [8 marks]

(4) Let $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Consider a subset of $C[a, b]$ (the collection of continuous functions on $[a, b]$)

$$S = \{\int_a^x f(t, g(t))dt, x \in [a, b] : g \in C[a, b] \text{ such that } g(t) \in [c, d] \text{ for all } t \in [a, b].\}$$

Show that S is precompact in $C[a, b]$ (under the metric $d(\phi_1, \phi_2) = \sup_{x\in[a,b]} |\phi_1(x) - \phi_2(x)|$). [5 marks]

(5) Compute (and justify) **one** of the following: [10 marks]

(i) $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx$,

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(6) If f is a nonnegative measurable function on $\mathbb{R}^n$ and $\int f(x)dx < \infty$, show that

$$\lim_{\alpha\to\infty} \int_{\{x : f(x) > \alpha\}} f(x)dx = 0.$$

[7 marks]

(7) Let $1 < p < \infty$ and $\{f_k\}$ be a bounded sequence of functions in $L^p(\mathbb{R}^n)$ (i.e., there exists $C > 0$ such that $\|f_k\|_p \leq C$). If $f_k \to f$ a.e., show that

$$\int f_k g dx \to \int f g dx$$

for all $g \in L^q(\mathbb{R}^n)$ where $1/q = (p - 1)/p$. [10 marks]

(8) Let $1 \leq p < \infty$ and $1/q = (p - 1)/p$. Let f be a measurable function on $[0, 1]$ such that

$$|\int_0^1 f g dx| \leq \|g\|_q \quad \text{for all step functions } g \text{ on } [0, 1] .$$

Show that $\|f\|_p \leq 1$. [10 marks]

(9) Prove or disprove **eight** of the following statements. [32 marks]

(a) If $f : [0,1] \to \mathbb{R}$ is a measurable function, then given any $\varepsilon > 0$, there exists a compact set $K \subset [0,1]$ with $|[0,1] \setminus K| < \varepsilon$ such that f is continuous on K.

(b) If $\{f_n\}$ is a nondecreasing sequence of Riemann integrable functions on $[0,1]$ that converges to 0 on $[0,1]$, then $\lim_{k\to\infty} \int f_k = 0$.

(c) If f is integrable on $[0,\pi]$, then $\lim_{n\to\infty} \int_0^\pi f(x)\cos nx dx = 0$.

(d) If f is a real function on $\mathbb{R}$ such that it is of bounded variation on $[a,b]$ for all $-\infty < a < b < \infty$, then f is continuous everywhere except countably many points.

(e) Let $\{f_n\}$ be a sequence of harmonic functions on the open unit disk in $\mathbb{R}^2$. If $f_n \to f$ uniformly on the open unit disk, then f is also harmonic on the open unit disk.

(f) Let U be a bounded open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$. If there exists a sequence of continuously differentiable functions $\{f_n\}$ that converges uniformly to f on U, then f is differentiable on U.

(g) If $f = u + iv$ (u and v are both real-valued functions) is an entire function such that $v(z) < 1$ for all z, then u must be a constant function.

(h) If f is an analytic function on an open connected set $\mathcal{D}$ (in the complex plane), then it is either a constant function or it will map open subsets of $\mathcal{D}$ to open sets.

(i) Let $\sum_{k=1}^{\infty} a_k$ be a convergent series. Then $\sum_{k=1}^{\infty} a_k \sin(k\pi x)$ converges if x is irrational.

(j) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function such that the Jacobian metric ∇f has nonzero determinant at the origin 0. If $f(0) = (1,0)$, then there exists $\varepsilon > 0$ such that for all $y \in \mathbb{R}^2$ with $|y - (1,0)| < \varepsilon$, the equation $f(x) = y$ has at least one solution.

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