NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

Qualification Examination

Analysis

August, 2010 — Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper comprises **THREE (3)** printed pages.
2. This paper consists of **EIGHT (8)** questions. Answer **ALL** of them. Marks for each question are not necessarily the same; marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Answer **all** *the questions in this paper*

Question 1 [10 marks]

(a) Consider the complex valued function $f(x+iy) = (x^2+y^2)+(xy)i$. Is f holomorphic on $\mathbf{C}$? If not, find a complex valued function $g(x+iy) = u(x,y)+iv(x,y)$ with $v(x,y) = 3x^2y - y^3$ such that $h := f+g$ is holomorphic on $\mathbf{C}$.

(b) Let h be the holomorphic function found in (a). Let Ω be the set $\{h(z) : |z| \leq 1\}$. Find the area of Ω.

Question 2 [10 marks]

Let f be a holomorphic function on $\mathbf{C}$. Suppose there exists a natural number n such that the limit, $\lim_{z\to\infty} \frac{f(z)}{z^n} = M$, exists for some constant M. Show that f is a polynomial with degree less than or equal to n.

Question 3 [15 marks]

Let u be a continuous, positive, integrable function on the interval $[0,\infty)$. Suppose there exist two positive constants a and b such that $\frac{du}{dt} \leq u(a+bu)$. Show that $\lim_{t\to\infty} u = 0$.

Question 4 [15 marks]

Let $\{f_n\}$ be a sequence of non-negative measurable functions on a measurable set E. If for any $\epsilon > 0$, $\sum_{n=1}^{\infty} |\{x \in E : f_n(x) > \epsilon\}| < \infty$, show that $\lim_{n\to\infty} f_n(x) = 0, a.e.$ on the set E. Is the converse also true? Justify your answer.

Question 5 [10 marks]

Let h be a holomorphic function from the unit disk into itself. If $h(0) = h'(0) = \cdots = h^{(k)}(0) = 0$ for some integer $k > 0$, show that $|h(z)| \leq |z|^{k+1}$ for all $|z| \leq 1$. Further show that there exists a z_0 with $|z_0| < 1$ such that $|h(z_0)| = |z_0|^{k+1}$ if and only if $h(z) = e^{i\theta} z^{k+1}$ for some constant $\theta \in [0, 2\pi)$.

Question 6 [10 marks]

Use two methods to show that

$$\lim_{n\to\infty} \int_0^1 \frac{n^{3/2}x}{1+n^2x^2} dx = 0.$$

Question 7 [15 marks]

Let $\{x_n\}$ be a strictly monotone increasing sequence with $x_0 \geq 0$. Show that the series

$$\sum_{n=1}^{\infty} (1 - \frac{x_n}{x_{n+1}})$$

converges if $\{x_n\}$ is bounded and diverges if $\{x_n\}$ is unbounded. (**Hint:** set $d_k = x_{k+1} - x_k$ for $k \geq 1$. Express the series in terms of d_n.)

Question 8 [15 marks]

Let $\varphi_1(x, y, z)$ and $\varphi_2(x, y, z)$ be continuously differentiable functions up to order 2 in the domain $\{(x, y, z) : x^2 + y^2 + z^2 < 4\}$. Show that (1) $\nabla\varphi_1 \times \nabla\varphi_2 = Curl(\varphi_1 \nabla\varphi_2)$ where $CurlV = \nabla \times V$ for any vector field V and $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ for any real valued function f; (2) the total flux $\oint_\Sigma A \cdot d\vec{S}$ of the vector $A = \nabla\varphi_1 \times \nabla\varphi_2$ through the surface $\Sigma := \{(x, y, z) | x^2 + y^2 + z^2 = 1\}$ is zero, where $d\vec{S}$ is the oriented surface area element.

END OF PAPER