Answer all questions. Each question carries 20 marks.

(1) Let G be a group, and let H be a subgroup of G of finite index n. Prove or disprove each of the following statements:
(a) If $a \in G$, then $a^n \in H$.
(b) If $a \in G$, then $a^k \in H$ for some $k \in \mathbb{Z}^+$, $1 \leq k \leq n$.

(2) Let $\alpha : V \to V$ be a linear operator on a finite-dimensional vector space V. Let W be the range of α, and let $\beta : W \to W$ be defined by $\beta(w) = \alpha(w)$ for all $w \in W$. Let $\chi_\alpha(x)$ and $\chi_\beta(x)$ denote the characteristic polynomial of α and β respectively. Prove that

$$\chi_\alpha(x) = x^k \chi_\beta(x),$$

where $k = \dim(V) - \dim(W)$.

(3) Let R be a ring with identity, and suppose that $I_1,\ I_2, \ldots, I_k$ are left ideals of R such that $R = \bigoplus_{j=1}^{k} I_j$ (as additive groups).
(a) Show that for each j, there exists $e_j \in I_j$ such that

$$ae_j = \begin{cases} a, & \text{if } a \in I_j; \\ 0, & \text{if } a \in I_k,\ k \neq j. \end{cases}$$

(b) Are the e_j's unique? Justify your answer.

(4) Let F be a field of positive characteristic p, and let $f(x) = x^p - x + a \in F[x]$. Let α be a root of $f(x)$ in some field extension of F. Show that $f(x)$ is a product of distinct linear polynomials over $F[\alpha]$.

(5) Let R be a principal ideal domain. Let M be a finitely generated left R-module. Suppose that $\{r \in R \mid rm = 0\ \forall m \in M\} = \mathfrak{p}^i$ for some prime ideal $\mathfrak{p}$ of R and some $i \in \mathbb{Z}^+$.
(a) Show that there exists $m_0 \in M$ such that $\{r \in R \mid rm_0 = 0\} = \mathfrak{p}^i$.
(b) Show that there exists a submodule N of M such that $M = Rm_0 \oplus N$.
(c) Deduce that $M = Rm_0 \oplus Rm_1 \oplus \cdots \oplus Rm_k$, where $\{r \in R \mid rm_j = 0\} = \mathfrak{p}^{i_j}$ for some $i_j \in \mathbb{Z}^+$ with $i_j \leq i$.
(You may assume that a submodule of a finitely generated module over a principal ideal domain is finitely generated. However, You may NOT assume any decomposition theorems of finitely generated modules over a principal ideal domain without proof.)