NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2009-2010

QF3101 Investment Instruments: Theory and Computation

April/May 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and **ONE** (1) appendix and comprises **SEVEN (7)** printed pages (including this cover page).
2. Answer **ALL** questions. The mark for each question is indicated at the beginning of the question.
3. Start your answer to each question on a new page.
4. This is a closed book examination. Use of help sheets is **not** allowed.
5. You may use a calculator. However, you should lay out systematically the various steps in the calculations.
6. Express all numerical answers up to **4 decimal places**.

Question 1 [*20 marks*]
Consider the two-factor model for asset i, $i = 1, 2$,

$$r_i = a_i + b_{1i} f_1 + b_{2i} f_2 + e_i$$

where $\mathrm{E}(e_i) = 0$, $\mathrm{cov}(e_i, f_1) = \mathrm{cov}(e_i, f_2) = 0$, and $\mathrm{cov}(e_i, e_j) = 0$ for $i \neq j$.

(i) Derive the formulae

$$\mathrm{var}(r_i) = b_{1i}^2 \mathrm{var}(f_1) + 2 b_{1i} b_{2i} \mathrm{cov}(f_1, f_2) + b_{2i}^2 \mathrm{var}(f_2) + \mathrm{var}(e_i).$$

(ii) Derive the linear system

$$\sum_{\ell=1}^{2} b_{\ell i} \ \mathrm{cov}(f_\ell, f_k) = \mathrm{cov}(r_i, f_k), \quad k = 1, 2.$$

(iii) The following estimates of the covariances have been obtained:

	Covariance with			
	Asset 1	Asset 2	Factor 1	Factor 2
Factor 1	0.15	0.08	0.49	0.32
Factor 2	0.03	0.13	0.32	0.81

Determine the factor loadings for both assets 1 and 2.

(iv) If $a_1 = 0.15$, $a_2 = 0.25$, $\mathrm{var}(e_1) = 0.04$ and $\mathrm{var}(e_2) = 0.01$, use the results obtained in (iii) to determine the variance of the rate of return for a portfolio which is equally weighted in both assets.

Question 2 [*20 marks*]

On April 21, 2010, a baker wishes to lock in the price of 80,000 pounds of butter which she wishes to take delivery on July 21, 2010. She has the choice of using either a forward contract or futures to achieve this. The storage cost of butter is \$0.12 per pound per year, payable monthly and in arrears. The current spot price of butter is \$1.50 per pound. Suppose the current 1-month, 2-month and 3-month spot rates are 0.50%, 0.52% and 0.58% (annualized, monthly compounding) respectively.

(i) Determine the key terms of the forward contract the baker could use to achieve her objective. Also determine the forward price per pound of butter for this contract.

(ii) Suppose one month later, on May 21, immediately after the storage cost payment, the spot price has risen to \$1.75 per pound, and the term structure is flat at 0.6% (annualized, monthly compounding). Determine the value of baker's position in this forward contract on May 21.

(iii) The CME has futures contract for butter with expiry months March, May, July, September, October and December, and each contract covers 40,000 pounds of butter. If the baker decides to use futures to achieve her objective, which of these contracts are likely candidates, and what criterion should she use to pick the correct one?

(iv) Suppose you are an arbitrageur, and that the expiry date of the July futures is July 21, 2010. If the futures price of the July contract on April 21 is higher than the forward price determined in (i), and that you are able to find a counter party for the necessary forward contract, describe an arbitrage strategy you could adopt on April 21.

Question 3 [*20 marks*]

(a) Suppose a 2-year interest rate swap (IRS-A) is to be configured on half-yearly in-arrear exchanges of a fixed rate payment for a floating level payment linked to a 6-month rate on a notional principal of \$500,000. The current 6-month, 12-month, 18-month and 24-month spot rates are 1.2%, 1.8%, 2.6% and 3.4% (annualized, semi-annual compounding) respectively.

(i) Determine the swap rate of IRS-A.

(ii) Suppose one year later, immediately after the 2nd swap exchange for IRS-A, a new 1-year interest rate swap (IRS-B) is initiated for half-yearly in-arrear exchanges with the same floating rate as for IRS-A. If the swap rate for IRS-B is lower than that for IRS-A at that point in time, what can you say about the value of IRS-A for the fixed rate receiver then?

(b) Company A wishes to borrow at a floating rate of interest for 5 years while Company B wishes to borrow at a fixed rate of interest for the same 5-year period. They have been quoted the following rates per annum:

	Fixed Rate	**Floating Rate**
Company A	2.5%	LIBOR+1.2%
Company B	3.5%	LIBOR+2.0%

Design a swap arrangement, with a bank acting as intermediary, that will produce an earning of 0.1% per annum for the bank, and equal rate savings for both companies.

Question 4 [*20 marks*]

(a) Suppose a Treasury bill (TB1) with 58 days to maturity is being quoted today at a discount yield of 0.72%, and another Treasury bill (TB2) with 148 days to maturity is being quoted today at a discount yield of 0.95%.

(i) Determine the dollar price and investment rate for Treasury bill TB2. Assume 365 days in the year concerned.

(ii) Determine the theoretical futures price (index basis) of a Treasury bill futures which expires 58 days from now.

(iii) For the Treasury bill futures which expires 58 days from now, if the actual futures price is lower than the theoretical price, briefly describe an arbitrage strategy which can be adopted by an arbitrageur.

(b) Suppose today is 16 March 2010, and the following table gives a summary of information on five Eurodollar futures with settlement prices recorded at the end of the day.

Contract	Settlement Price	Contract Expiry Date	#Days from 16-March to Expiry Date
Mar 10	99.685	16-Mar-2010	0
Jun 10	99.615	14-Jun-2010	90
Sep 10	99.390	13-Sep-2010	181
Dec 10	99.050	13-Dec-2010	272
Mar 11	98.650	14-Mar-2011	363

(i) If a 9-month loan, maturing on 13 Dec, is initiated today on 9-month LIBOR plus 0.5%, determine the actual loan rate (actual/360).

(ii) Determine the fair fixed rate for a 6x12 forward rate agreement initiated today.

Question 5 [*20 marks*]

(a) A portfolio has been mapped to a portfolio of three risk factors, namely factor 1, factor 2 and factor 3. Dollar exposures, daily volatilities and the correlation matrix for the factors' returns are given below. If the factors' daily returns are normally distributed, determine the 5-day 95% VaR of this portfolio.

Position	dollar exposure	daily volatility	Correlation with factor 1	factor 2	factor 3
factor 1	\$10,000	0.1%	1	0.12	0.25
factor 2	\$15,000	0.2%	0.12	1	-0.73
factor 3	$-\$25,000$	0.8%	0.25	-0.73	1

(b) A bond trading book holds a position on a zero coupon bond with a face value of \$500,000. This zero will mature in 1.25 years.

(i) Given the following information, use cash flow mapping to map this position to the adjacent standard time vertices of 1 and 2 years.

Standard Maturity	Yield (%)	Price Volatility (daily %)	Correlation Matrix, R	
			2 years	3 years
1 year	3.28	0.15	1	0.96
2 years	3.19	0.22	0.96	1

(ii) Hence determine the 1-day 95% VaR of this zero-coupon bond position.

END OF PAPER

Appendix: QF3101 Formula Sheet (Two pages)

For asset i, we have

- Capital Asset Pricing Model (CAPM): $\overline{r}_i - r_f = \beta_i(\overline{r}_M - r_f)$, where $\beta_i = \sigma_{iM}/\sigma_M^2$.
- Systematic and Firm-Specific Risk: $\sigma_i{}^2 = \beta_i^2\sigma_M^2 + \text{var}(\epsilon_i)$

Single Factor Models For asset i, we have $r_i = a_i + b_i f + e_i, i = 1, 2, \ldots, n$, and

$$\begin{aligned} \bar{r}_i &= a_i + b_i\bar{f}, & \sigma_i^2 &= b_i^2\sigma_f^2 + \sigma_{e_i}^2 \\ \sigma_{ij} &= b_i b_j \sigma_f^2, \qquad i \neq j, & b_i &= \text{cov}(r_i, f)/\sigma_f^2 \\ R^2 &= (\text{var}(r_i) - \text{var}(e_i))/\text{var}(r_i). \end{aligned}$$

For a portfolio of assets following single-factor models: $\sigma_p^2 = b^2\sigma_f^2 + \sigma_e^2$ where $b = \sum_{i=1}^n w_i b_i$, $e = \sum_{i=1}^n w_i e_i$ and $\sigma_e^2 = \sum_{i=1}^n w_i^2 \sigma_{e_i}^2$.

Two-Factor Models For asset i, we have $r_i = a_i + b_{1i}f_1 + b_{2i}f_2 + e_i, i = 1, 2, \ldots, n$, and

$$\begin{aligned} \overline{r}_i &= a_i + b_{1i}\overline{f}_1 + b_{2i}\overline{f}_2, \\ \text{cov}(r_i, r_j) &= b_{1i}b_{1j}\text{var}(f_1) + (b_{1i}b_{2j} + b_{2i}b_{1j})\text{cov}(f_1, f_2) + b_{2i}b_{2j}\text{var}(f_2) + \text{cov}(e_i, e_j) \\ \text{cov}(r_i, f_1) &= b_{1i}\,\text{var}(f_1) + b_{2i}\,\text{cov}(f_1, f_2) \\ \text{cov}(r_i, f_2) &= b_{1i}\,\text{cov}(f_1, f_2) + b_{2i}\,\text{var}(f_2). \end{aligned}$$

Arbitrage Pricing Theory For asset i on m-factor model: $r_i = a_i + \sum_{j=1}^m b_{ji} f_j + e_i$, APT implies that
$\overline{r}_i = \lambda_0 + \sum_{j=1}^m b_{ji}\lambda_j$, where λ_0 is the risk-free rate, and λ_j is the factor price for factor j, $j = 1, \ldots, n$.

Forward Price Formulae

- $F_0 = S_0/d(0,T)$,
- $F_0 = (S_0 - I)/d(0,T)$,
- $F_0 = S_0 e^{(r-q)T}$
- $F_0 = \dfrac{S_0}{d(0,M)} + \displaystyle\sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)}$
- $F_0 = \dfrac{S_0}{d(0,M)} + \displaystyle\sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)} - \sum_{k=0}^{M-1} \frac{y}{d(k,M)}$

Interest Rate Swap The initial value of an interest rate swap with maturity T to the company receiving floating and paying fixed on notional principal N is

$$V = \left(1 - d(0,T) - r\sum_{i=1}^{M} d(0,t_i)\right) N.$$

The value of swap with time t_1 to the next exchange for the company receiving fixed and paying floating is

$$V = Nd(0,T) + \sum_{i=1}^{n} k\ d(0,t_i) - (N + f_1)d(0,t_1).$$

Currency Swap The value of swap for the company paying currency 1 and receiving currency 2 is $V = SP_2 - P_1$, where S is the spot exchange rate, and P_i, $i = 1, 2$, are the values of the bonds in currencies 1 and 2 respectively.

Hedge Basis $b_i = S_i - F_i$, Effective price for hedging $F_1 + b_2$,
Minimum variance hedge ratio $\beta^* = \dfrac{\text{cov}(S_T, F_T)}{\text{var}(F_T)}$.

Changing beta of a Portfolio No. of index contracts= $(\beta_1 - \beta_2)\dfrac{S}{m\,I_0}$.

T-bill discount yield= $\dfrac{100 - P}{100} \times \dfrac{360}{t}$, Investment rate (less than half year) $i = \dfrac{100 - P}{P} \times \dfrac{y}{t}$.

For more than one half-year to maturity, solve $(t/(2y) - 0.25)i^2 + (t/y)i + (P - 100)/P = 0$.

Full price of a bond= $\displaystyle\sum_{i=0}^{n-1} \frac{C}{\left(1 + \frac{r}{m}\right)^{i+w}} + \frac{F}{\left(1 + \frac{r}{m}\right)^{w+n-1}}$, accrued interest= $(1 - w)C$.

Add-on yield $= \dfrac{\textsf{Interest}}{\textsf{Principal}} \times \dfrac{360}{\textsf{Days to maturity}}$.

Eurodollar Futures Price Index Price= 100−(annualized add-on yield for the 3-month period).

FRA payoff to buyer $(y - X) \times N \times \dfrac{m}{360} \times \dfrac{1}{1 + y(m/360)}$.

Forward rate for $[T, T + m]$ $f_L(T, T + m) = \left(\dfrac{d_L(T)}{d_L(T + m)} - 1\right) \Big/ (m/360)$.

PV of FRA $[f_L(T, T + m) - X] \times (m/360) \times d_L(T + m)$.

Currency Forward Forward Price $F_0 = x_0\, d(r_f, T)/d(r_l, T)$, r_f and r_l are foreign and local interest rates respectively.

Value-at-Risk
For single position, relative VaR: $\text{VaR}_r = \alpha\sigma W_0$, absolute VaR: $\text{VaR}_a = (\alpha\sigma - \mu)W_0$.

$\alpha = 1.65$ for 95% confidence level, and $\alpha = 2.33$ for 99% confidence level.

Square root of time rule N-day VaR= 1-day VaR$\times\sqrt{N}$.

Delta-Normal method Portfolio VaR is $\text{VaR}_p = \alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}$.

Cash Flow Mapping For mapping zero at t to zeros at standard maturities p and q with $p < t < q$.
Solve quadratic for alpha: $(\sigma_p^2 + \sigma_q^2 - 2\rho_{p,q}\sigma_p\sigma_q)\alpha^2 + (2\rho_{p,q}\sigma_p\sigma_q - 2\sigma_q^2)\alpha + (\sigma_q^2 - \sigma_t^2) = 0$.