NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2009-2010

MA2214 Combinatorial Analysis

April/May 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly on the space provided below.** This booklet will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper contains **NINE (9)** questions and comprises **ELEVEN (11)** printed pages.

3. Answer **ALL** questions.

4. Calculators may be used. However, various steps in the calculations should be laid out systematically.

CANDIDATE'S MATRICULATION NUMBER :

FOR OFFICIAL USE ONLY:

1	2	3	4	5	6	7	8	9	Total

*Answer **all** questions.*

Question 1 [5 marks]

Suppose that there are m distinct circles $T_1, \ldots, T_m$. Let $n = 1 + 2 + \ldots + m$. Find the number of ways to arrange n people such that there are exactly i people around the circle T_i for each $1 \leq i \leq m$.

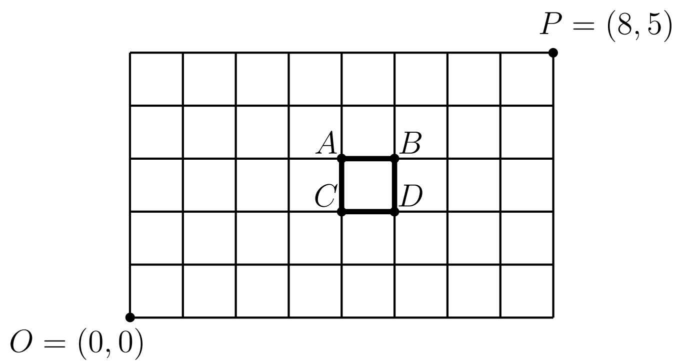
Question 2 [15 marks]

(i) [9 marks] How many positive integers strictly less than 2010 are multiples of 3 or 4 but not 5?

(ii) [6 marks] How many positive integers strictly less than 3001 are divisible by exactly three of the prime numbers $2, 3, 7, 11$?

Question 3 [10 marks]

Consider the following route system.



Suppose that the loop $ABDC$ has been deleted. Find the number of shortest routes from O to P.

Question 4 [15 marks]

Let ϕ be the Euler function. Find all positive integers $n \geq 2$ such that $\phi(n) = 20$.

Question 5 [10 marks]

Let a_n be the number of non-negative integer solutions to the equation $x_1+2x_2+x_3+2x_4 = n$ such that $x_1 \geq 10$ and $1 \leq x_4 \leq 10$. Show that the ordinary generating function for the sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ is

$$x^{12}\frac{1-x^{20}}{(1-x)^4(1+x)^2}.$$

Question 6 [10 marks]

Solve the recurrence relation $a_n - 7a_{n-1} + 12a_{n-2} = 5^n - 3^n$ for $n \geq 2$ and $a_0 = a_1 = 1$.

Question 7 [10 marks]

A die consists of six faces with each face representing precisely one of the numbers $1, 2, 3, 4, 5, 6$. Suppose that n such dice are rolled for some positive integer n. The number on the upper face of each die is noted and the sum of these numbers are computed. Show that the number of ways to get an even sum is

$$\sum_{m=0}^{N} H^3_{2m} H^3_{n-2m}$$

where $H^r_s = \binom{r+s-1}{s}$ and N is the largest integer such that $2N \leq n$.

Question 8 [10 marks]

(i) [5 marks] Show that the number of ways to choose two (unordered) subsets S, T of $\{1, 2, \ldots, n\}$ such that $S \cap T = \varnothing$ is 3^n.

(ii) [5 marks] Show that the number of ways to choose two (unordered) subsets S, T of $\{1, 2, \ldots, n\}$ such that $S \cap T \neq \varnothing$ is $4^n - 3^n$.

Question 9 [15 marks]

Consider a rectangular wall of dimension $2 \times n$ where 2 is the height and n is the length. Fix a non-negative integer m. Suppose that we have m distinct colours $c_1, \ldots, c_m$. We wish to pave the wall using m types of tiles $t_1, \ldots, t_m$ where each of them is of dimension 1×1 and with single colour c_i for some $1 \leq i \leq m$. Two tiles are adjacent if they share a common edge. Let $a(m)_n$ denote the number of ways to pave the $2 \times n$ wall such that there is no two adjacent tiles that are in the same colour. We assume that the wall is upright so that, for instance, the following two tilings are different.

$$T_1 = \begin{array}{|c|c|c|} \hline t_2 & t_1 & t_2 \\ \hline t_1 & t_2 & t_1 \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c|} \hline t_1 & t_2 & t_1 \\ \hline t_2 & t_1 & t_2 \\ \hline \end{array}$$

(i) Find a recurrence relation for $a(m)_n$ for $n \geq 2$.

(ii) Find a formula for $a(m)_n$ in terms of m and n assuming that

$$a(m)_0 = \frac{m(m-1)}{m^2 - 3m + 3}.$$

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