NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2009-2010

MA2213 Numerical Analysis I

April 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring one piece of A4-sized two-sided help sheet into the examination room.

2. This examination paper contains a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

3. Answer **ALL** questions in **Section A**. Each question in Section A carries 20 marks.

4. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1. [20 marks]

(a) Let

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 1 \\ 2 & 1 & 3 & -1 \\ 1 & -2 & 2 & -2 \end{bmatrix}.$$

Does A have an LU factorization where L is lower triangular with 1's on its diagonal and U is upper triangular? If not, determine if there is a permutation matrix P such that $PA = LU$. Find the matrices L, U, and P.

(b) Give exact way to avoid loss of significance in the following computations.

(1) $\log(x+1) - \log(x)$ large x;

(2) $\tan(x) - \tan(y)$ $x \approx y$;

(3) $\frac{1-\cos(x)}{x^2}$ $x \approx 0$;

(4) $\sqrt[3]{1+x} - 1$ $x \approx 0$.

Question 2. [20 marks]

(a) For a given function $f(x)$ and two distinct points x_0 and x_1, find a polynomial $p(x)$ of degree ≤ 2 that is in the form

$$p(x) = f(x_0)l_0(x) + f'(x_0)l_1(x) + f'(x_1)l_2(x)$$

and satisfies

$$p(x_0) = f(x_0), \qquad p'(x_0) = f'(x_0), \qquad p'(x_1) = f'(x_1).$$

(b) Find $S'(0)$ and $S'(3)$ for the cubic spline

$$S(x) = \begin{cases} 3 + b_1 x + x^3 & \text{on } [0,1] \\ 1 + b_2(x-1) + 3(x-1)^2 - 2(x-1)^3 & \text{on } [1,3] \end{cases}.$$

Question 3. [20 marks]

(a) Determine constants a, b, c, d, and e that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(0) + cf(1) + df'(-1) + ef'(1)$$

that has the highest degree of precision.

(b) The error term for Composite Trapezoidal rule T_n for approximating $I = \int_a^b f(x)dx$ can be rewritten as

$$I - T_n = -\frac{h^2}{12}[f'(b) - f'(a)] + O(h^4)$$

where $h = \frac{b-a}{n}$ and n is the number of subintervals. The Composite Midpoint rule M_n is obtained by applying the Midpoint rule to each of the n subintervals and we have

$$I - M_n = \frac{h^2}{24}[f'(b) - f'(a)] + O(h^4).$$

Using these results, obtain a new numerical integration formula S_n combining T_n and M_n, with error $O(h^4)$. Show that the error for S_n is given by

$$I - S_n = -\frac{b-a}{2880}h^4 f^{(4)}(\mu).$$

SECTION B

Answer not more than **two** *questions from this section. Section B carries a total of 40 marks.*

Question 4. [20 marks]

(i) Suppose that g is continuously differentiable on some interval (c, d) that contains the fixed point p of g. Show that if $|g'(p)| < 1$, then there exists a $\delta > 0$ such that if $|p_0 - p| \leq \delta$, then the fixed-point iteration converges.

(ii) The iteration $p_{n+1} = 2 - (1 + c)p_n + cp_n^3$ will converge to the fixed point $p = 1$ for some values of c provided p_0 is chosen sufficiently close to p. Find the values of c for which this is true. For what value of c will the convergence be quadratic?

Question 5. [20 marks]

(a) Use the cubic Hermite interpolation polynomial that interpolates $f(x)$ at $x = a, b$ to obtain a formula to approximate the integral $\int_a^b f(x)dx$. Then derive the error term for the approximation.

(b) Let $P_3(x)$ be a degree 3 polynomial, and let $P_2(x)$ be its interpolating polynomial at the three points $x = a, a+h$, and $a+2h$. If $\int_a^{a+2h} P_2(x)dx = A$, find the value of $\int_a^{a+2h} P_3(x)dx$ in terms of a, h, and A.

Question 6. [20 marks]

Find the least squares polynomial approximation of the form $P(x) = ax^2 + b$ on the interval $[0, 1]$ for the function

$$f(x) = \sqrt{1 - x^2}$$

and then use the result to approximate the value of $\sqrt{21}$.

END OF PAPER