NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2009-2010

**MA2108 Mathematical Analysis I**

April 2010 — Time allowed : 2 hours

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**INSTRUCTIONS TO CANDIDATES**

1. This is a closed book examination. Each student is allowed to bring one piece of A4-sized two-sided help sheet into the examination room.

2. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **NINE (9)** questions and comprises **FIVE (5)** printed pages.

3. Answer **ALL** questions in **Section A**. Section A carries a total of 70 marks.

4. Answer not more than **TWO (2)** questions from **Section B**. Section B carries a total of 30 marks.

5. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

## SECTION A

*Answer* **all** *the questions in this section. Section A carries a total of 70 marks.*

**Question 1.**

Let

$$x_1 = 1, \qquad x_{n+1} = \frac{1}{5}\left(x_n^2 + 6\right), \ \forall n \in \mathbb{N}.$$

(i) Prove that $x_n \leq 2$ for all $n \in \mathbb{N}$. [3 marks]

(ii) Prove that $(x_n)$ converges and find its limit. [7 marks]

**Question 2.**

(a) Determine whether the following series converge or diverge. Justify your answers.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n + \sqrt{n} + 1}$. [4 marks]

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}\left(1 + \frac{1}{3n}\right)^{6n^2}$. [4 marks]

(b) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$. [6 marks]

(c) Let $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ be two convergent series with positive terms, that is, $a_n > 0$ and $b_n > 0$ for all $n \in \mathbb{N}$. Prove that $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ also converges. [6 marks]

**Question 3.**

(a) Use the $\varepsilon - \delta$ definition of limit to prove that

$$\lim_{x\to 0} \frac{(2x+1)(x-2)}{3x+1} = -2.$$

[7 marks]

(b) In each part, either evaluate the limit or show that the limit does not exist. Here $[x]$ denotes the greatest integer less than or equal to $x$.

(i) $\lim_{x\to 0}(x^2+x+1)\sin\left(\frac{3}{x}\right)$. [4 marks]

(ii) $\lim_{x\to 0^+} \frac{x}{2}\left[\frac{6}{x}\right]$. [4 marks]

**Question 4.**

Let $a \in \mathbb{R}$. Suppose that the functions $f$ and $g$ have the following properties:

(i) There exist $M > 0$ and $h > 0$ such that

$$|f(x)| \leq M \quad \text{for all } x \text{ satisfying } 0 < |x-a| < h.$$

(ii) $\lim_{x\to a} g(x) = 0$.

Prove that $\lim_{x\to a} f(x)g(x) = 0$. [7 marks]

**Question 5.**

Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x & \text{if } x \text{ is rational} \\ 3x-8 & \text{if } x \text{ is irrational.} \end{cases}$$

Find all the point(s) at which $f$ is continuous. Justify your claim. [8 marks]

**Question 6.**

Suppose that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous on $\mathbb{R}$, and

$$|f(x)| \leq \frac{1}{2}, \quad |g(x)| \leq 2 \quad \text{for all } x \in \mathbb{R}.$$

Define the function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = f(x)g(x) \quad x \in \mathbb{R}.$$

Prove that $F$ is also uniformly continuous on $\mathbb{R}$. [10 marks]

**SECTION B**

*Answer not more than* **two** *questions from this section. Section B carries a total of 30 marks.*

**Question 7.**

(a) Let $(x_n)$ and $(y_n)$ be bounded sequences. Prove that

$$\limsup(x_n - y_n) \leq \limsup(x_n) - \liminf(y_n).$$

[7 marks]

(b) Suppose that $(b_n)$ is a sequence of positive numbers and the series $\sum_{n=1}^{\infty} b_n$ is divergent. For each $n \in \mathbb{N}$, let

$$S_n = \sum_{k=1}^{n} b_k.$$

(i) Prove that for each $n > 1$,

$$\frac{b_n}{S_n^2} < \frac{1}{S_{n-1}} - \frac{1}{S_n}.$$

[3 marks]

(ii) Is the series $\sum_{n=1}^{\infty} \frac{b_n}{S_n^2}$ convergent? Justify your answer. [5 marks]

**Question 8.**

(a) Let the function $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\infty} f(x) = L$ where $L \in \mathbb{R}$. Prove that if $(x_n)$ is a sequence such that $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty} f(x_n) = L$.

[7 marks]

(b) Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ has the following property: for each $\varepsilon > 0$, there exists $M \in \mathbb{R}$ such that

$$|g(x) - g(x')| < \varepsilon \quad \text{whenever } x > M \text{ and } x' > M.$$

Prove that $\lim_{x\to\infty} g(x)$ exists. [8 marks]

**Question 9.**

(a) Suppose that the function $f : (0,1) \to \mathbb{R}$ is continuous on $(0,1)$ and $x_1, x_2, ...., x_n$ are in $(0,1)$. Prove that there exists $c \in (0,1)$ such that

$$f(c) = \frac{1}{n}(f(x_1) + \cdots + f(x_n)).$$

[7 marks]

(b) Let the function $g : [0,\infty) \to \mathbb{R}$ be uniformly continuous on $[0,\infty)$ and $g(0) = 0$. Prove that there exists $C > 0$ such that

$$|g(x)| < 1 + Cx \quad \text{for all } x > 0.$$

[8 marks]

**END OF PAPER**