NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2009-2010

MA2101 Linear Algebra II

April 2010 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections. It contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.

3. Answer not more than **TWO (2)** questions in **Section B**. Each question in Section B carries 20 marks.

4. Calculators may be used. However, various steps in the calculations should be laid out systematically.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [15 Marks]

Let V be a real vector space with basis $B = \{\boldsymbol{v_1},\ \boldsymbol{v_2},\ \boldsymbol{v_3},\ \boldsymbol{v_4}\}$.

Define $W_1 = \text{span}\{\boldsymbol{v_1} + \boldsymbol{v_2},\ \boldsymbol{v_2} + \boldsymbol{v_3},\ \boldsymbol{v_3} + \boldsymbol{v_4},\ \boldsymbol{v_1} + \boldsymbol{v_4}\}$ and $W_2 = \text{span}\{\boldsymbol{v_4}\}$.

(a) Find the dimensions of W_1, W_2, $W_1 \cap W_2$ and $W_1 + W_2$.

(b) Is $V = W_1 \oplus W_2$? Justify your answer.

(c) Give an example of a subspace of V such that $V = W_1 \oplus U$ and $U \neq W_2$.

Question 2 [15 Marks]

Let $T : V \to W$ be a linear transformation. For $X \subseteq V$, define $T(X) = \{T(\boldsymbol{u}) \mid \boldsymbol{u} \in X\}$.

(a) Let X be a subspace of V. Show that $T(X)$ is a subspace of W.

(b) Let $V = W = \mathbb{R}^3$ and $T((x, y, z)) = (x - y,\ y - z,\ z - x)$ for $(x, y, z) \in \mathbb{R}^3$.

  (i) Find nullity(T) and rank(T).

  (ii) Let $X = \{(x, x, z) \mid x, z \in \mathbb{R}\}$. Write down $T(X)$ explicitly and find its dimension.

Question 3 [15 Marks]

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 1 & -2 & 1 \\ -1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \in \mathcal{M}_{4\times4}(\mathbb{R})$.

(a) Compute the characteristic polynomial of $\boldsymbol{A}$.

(b) Determine the dimension of the eigenspace of $\boldsymbol{A}$ associated with the eigenvalue 2.

(c) Find a Jordan canonical form for $\boldsymbol{A}$.

Question 4 [15 Marks]

Suppose $\mathcal{M}_{n\times n}(\mathbb{C})$ is equipped with the usual inner product.

(a) Let T be the linear operator on $\mathcal{M}_{n\times n}(\mathbb{C})$ defined by

$$T(\boldsymbol{X}) = \boldsymbol{BX} \quad \text{for } \boldsymbol{X} \in \mathcal{M}_{n\times n}(\mathbb{C})$$

where $\boldsymbol{B}$ is an $n \times n$ complex matrix.

(i) Find the adjoint of T.

(ii) Prove that T is unitarily diagonalizable if and only if $\boldsymbol{B}$ is normal.

(b) Suppose T_1, T_2, T_3 are linear operators on $\mathcal{M}_{2\times 2}(\mathbb{C})$ defined by

$$T_j(\boldsymbol{X}) = \boldsymbol{B_j X} \quad \text{for } \boldsymbol{X} \in \mathcal{M}_{2\times 2}(\mathbb{C})$$

where $\boldsymbol{B_1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\boldsymbol{B_2} = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$ and $\boldsymbol{B_3} = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}$.

For $j = 1, 2, 3$, determine whether T_j is unitarily diagonalizable.

SECTION B

Answer not more than **two** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 Marks] (All vectors of $\mathbb{R}^n$ and $\mathbb{R}^m$ in this question are column vectors.)

Let $\boldsymbol{A}$ be an $m \times n$ real matrix and $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k} \in \mathbb{R}^n$.

(a) Show that if $\boldsymbol{Av_1}$, $\boldsymbol{Av_2}$, $\ldots$, $\boldsymbol{Av_k}$ are linearly independent vectors in $\mathbb{R}^m$, then $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\ldots$, $\boldsymbol{v_k}$ are linearly independent vectors in $\mathbb{R}^n$.

Let N be the nullspace of $\boldsymbol{A}$, i.e. $N = \{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{Au} = \boldsymbol{0}\}$.

(b) Prove that $\boldsymbol{Av_1}$, $\boldsymbol{Av_2}$, $\ldots$, $\boldsymbol{Av_k}$ are linearly independent vectors in $\mathbb{R}^m$ if and only if $N + \boldsymbol{v_1}$, $N + \boldsymbol{v_2}$, $\ldots$, $N + \boldsymbol{v_k}$ are linearly independent vectors in $\mathbb{R}^n/N$.

(c) Give an example of a non-zero 2×2 real matrix $\boldsymbol{A}$ and $\boldsymbol{v_1}, \boldsymbol{v}_2 \in \mathbb{R}^2$ such that $\boldsymbol{v_1}$, $\boldsymbol{v}_2$ are linearly independent but $\boldsymbol{Av_1}$, $\boldsymbol{Av}_2$ are linearly dependent. With your example, write down a basis for $\{\boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^2\}$ and a basis for $\mathbb{R}^2/N$.

Question 6 [20 Marks]

Let V be a finite dimensional vector space.

(a) Let S and T be linear operators on V and W a subspace of V. Suppose W is both S-invariant and T-invariant. Show that W is $(S \circ T)$-invariant and $(S \circ T)|_W = (S|_W) \circ (T|_W)$.

(b) Let S and T be linear operators on V such that $S \circ T = T \circ S$. Suppose E_λ is an eigenspace of T associated with an eigenvalue λ. Prove that E_λ is S-invariant.

(c) Let $T_1, T_2, \ldots, T_n$ be diagonalizable linear operators on V such that $T_i \circ T_j = T_j \circ T_i$ for all $i, j \in \{1, 2, \ldots, n\}$. Prove that there exists an ordered basis B for V such that $[T_1]_B, [T_2]_B, \ldots, [T_n]_B$ are diagonal matrices.

(Hint: If T is a diagonalizable linear operator on V and W is a T-invariant subspace of V, then $T|_W$ is a diagonalizable linear operator on W.)

Question 7 [20 Marks]

(a) Let P be a linear operator on a vector space V such that $P^2 = P$. Prove that $V = \mathrm{R}(P) \oplus \mathrm{Ker}(P)$.
(Warning: V may be infinite dimensional.)

(b) Give an example of a linear operator P on $V = \mathbb{R}^2$ such that $P^2 = P$ but $P \neq O_V$ and $P \neq I_V$, where O_V is the zero operator on V and I_V is the identity operator on V. With your example P, write down $\mathrm{R}(P)$ and $\mathrm{Ker}(P)$ explicitly.

(c) Suppose V is an inner product space. Let P be a linear operator on V such that $P^2 = P$ and $\langle P(\boldsymbol{u}), \boldsymbol{u} - P(\boldsymbol{u}) \rangle = 0$ for all $\boldsymbol{u} \in V$.
Prove that $\mathrm{Ker}(P) = \mathrm{R}(P)^\perp$.

[END OF PAPER]