NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2009 – 2010

MA1521 Calculus For Computing

April 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **TEN (10)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** the questions. Each question carries 10 marks.
3. Non-programmable calculators may be used. However, you should lay out systematically the various steps in your calculations.

Question 1

(a) Find the directional derivative of $f(x, y) = ye^{2x+4y}$ at the point $(-2, 1)$ in the direction of $-2\mathbf{i} + \frac{3}{2}\mathbf{j}$.

(b) Find a Cartesian equation of the tangent plane to the surface $z = xy(x + y)$ at the point $(1, 1, 2)$.

Question 2

Let $f(x) = \dfrac{x^2}{3 - 4x}$. It is given that $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ for $-\frac{3}{4} < x < \frac{3}{4}$.

(i) Find the exact value of $f^{(2010)}(0)$.

(ii) Evaluate the series $\displaystyle\sum_{n=3}^{\infty} \frac{n}{c_n}$.

Question 3

The equation $2\ln(x - 2) + x = 0$ has a unique positive real root, α.

(i) Use the Intermediate Value Theorem to show that $2 < \alpha < 3$.

(ii) Show that $\alpha_0 = 3$ is *not* a suitable initial approximation for the Newton-Raphson method.

(iii) Taking α_0 to be 2.5, use the Newton-Raphson method to find α to three significant figures.

(iv) Determine whether your answer in (iii) is an overestimate or an underestimate of α.

Question 4

Evaluate the following limits.

(a) $\displaystyle\lim_{x\to\infty} (1 + 2e^{-x})^{3e^x}$

(b) $\displaystyle\lim_{x\to 0^+} \frac{\int_0^x t\sin t\, dt}{\int_0^{2x} \tan(2t^2)\, dt}$

Question 5

Solve the differential equation $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = 0$, given that $y = 3$ and $\frac{dy}{dx} = 14$ when $x = 0$.

Question 6

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\sum_{n=4}^{\infty} \left(\frac{\sqrt{6}}{n}\right)^n n!$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n + 2\ln n + 3}$

(c) $\sum_{n=1}^{\infty} \frac{\cos(\frac{1}{\sqrt{n}})}{\sqrt{n^{1/n}}}$

Question 7

(a) Show that the series $\sum_{n=1}^{\infty} \frac{\sqrt{2n-1} + \sqrt{2n+1}}{1 + 2 + 3 + \cdots + n}$ is convergent.

(b) Show that the series $\sum_{n=1}^{\infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}}{1 + 2 + 3 + \cdots + n}$ is divergent.

Question 8

Given that $I_n = \int_0^1 x^n e^{3x}\, dx$ where n is a positive integer, use integration by parts to show that for $n \geq 1$,

$$I_n = \frac{1}{3}(e^3 - nI_{n-1}).$$

Evaluate I_3 and hence, deduce the exact value of

$$\int_0^1 xe^{3\sqrt{x}}\, dx.$$

Question 9

Let $p, q > 1$ be two numbers such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and let $f(x, y) = x^{1/p}y^{1/q}$ where $x \geq 0$ and $y \geq 0$. Use the method of Lagrange multipliers to maximize $f(x, y)$ subject to the constraint $\dfrac{x}{p} + \dfrac{y}{q} = c$, for some positive constant c. Hence, prove the Young's inequality

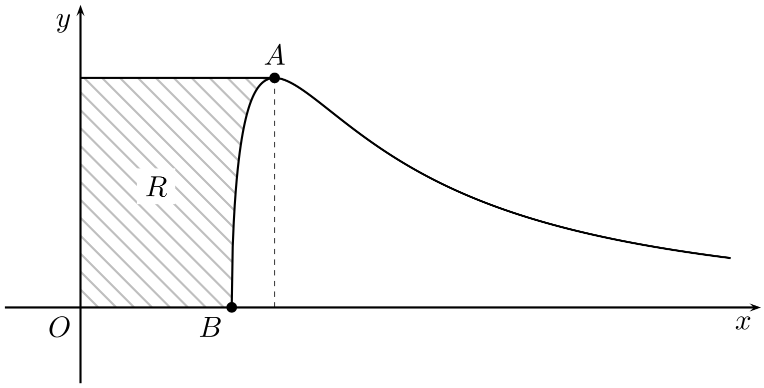
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for all $a \geq 0$ and $b \geq 0$.

Let $\{x_n\}_{n=1}^{N}$ and $\{y_n\}_{n=1}^{N}$ be two finite sequences of positive real numbers. Use the above Young's inequality to prove that

$$\sum_{n=1}^{N} \sqrt{x_n y_n} \leq \sqrt{\sum_{n=1}^{N} x_n \sum_{n=1}^{N} y_n}.$$

Question 10



The diagram shows part of the curve whose equation is $y = \dfrac{\sqrt{\ln x}}{x^2}$. A is the stationary point on the curve and B is the point at which the curve meets the x-axis.

(i) Calculate the coordinates of A and B.

(ii) Use the trapezoidal rule with 5 ordinates to obtain an approximation to the area bounded by the curve and the x-axis for $1 \leq x \leq 2$.

The region R is bounded by the axes, the curve and the horizontal line that passes through the point A.

(iii) Calculate the exact volume of the solid formed by rotating the region R completely about the y-axis.