NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2009-2010

MA1104 Multivariable Calculus

April 2010 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring two pieces of A4-sized two-sided help sheets into the examination room.

2. This examination paper consists of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

3. Answer **ALL** questions. Marks for each question are indicated at the end of the question.

4. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

Answer **ALL** *questions.*

Question 1

(a) Find an equation of the tangent plane to the hyperboloid given by

$$z^2 - 2x^2 - 2y^2 = 12$$

at the point $(1, -1, 4)$.

[10 *marks*]

(b) Find all critical points of the following function and classify them:

$$f(x, y) = 6xy^2 - 2x^3 - 3y^4.$$

[10 *marks*]

Question 2

(a) Let

$$f(x,y) = \begin{cases} (x^3 + y^3)\cos\left(\frac{1}{x^3+y^3}\right) & \text{if } \ (x,y) \neq (0,0) \\ 0 & \text{if } \ (x,y) = (0,0). \end{cases}$$

(i) Show that $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

(ii) Is $f(x, y)$ differentiable at $(0, 0)$? Justify your answer.

(iii) Is $f_x(x, y)$ continuous at $(0, 0)$? Justify your answer.

[8 *marks*]

(b) Evaluate the following iterated integral:

$$\int_0^2 \int_{x/2}^1 \sin(y^2)\, dy\, dx.$$

[8 *marks*]

Question 3

(a) Let a be positive real number. If f is continuous, show that

$$\int_0^a \int_0^y \int_0^z f(x)\,dx\,dz\,dy = \frac{1}{2}\int_0^a (a-x)^2 f(x)\,dx.$$

Hint: Identify the solid and change the order of integration. [5 *marks*]

(b) Let W be the solid bounded below by the upper hemisphere of $x^2+y^2+z^2=6$ and bounded above by the paraboloid $z=4-x^2-y^2$. Find the volume of W.

[8 *marks*]

(c) Let $D=\{(x,y) : 10 \le xy \le 20,\ 20 \le x^2y \le 40\}$. Using Change of Variables, evaluate the following integral

$$\iint_D e^{xy}\,dA.$$

Hint: Choose $u=xy$, $v=x^2y$. [7 *marks*]

Question 4

(a) The production of a company is given by the Cobb-Douglas function $P(L,K) = 200L^{2/3}K^{1/3}$ where $L \ge 0$ is the labor and $K \ge 0$ is the capital. However, cost constraints on the business forces $2L+5K \le 150$. Find the values of L and K which maximize the production. Explain why your answers give the maximum production.

[7 *marks*]

(b) Let $\mathbf{F}(x,y) = \langle 10x^4 - 2xy^3, -3x^2y^2\rangle$. Show that $\mathbf{F}$ is conservative. Hence or otherwise, evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ from $(0,0)$ to $(2,1)$ along the path $x^4-6xy^3=4y^2$.

[6 *marks*]

(c) Use a line integral to find the area of the region (on the xy-plane) bounded by the curve

$$x(t)=\frac{1}{2}\sin 2t, \quad y(t)=\sin t, \quad 0 \le t \le 2\pi.$$

Hint: $\oint_C x\,dy = -\oint_C y\,dx$.

[4 *marks*]

Question 5

Let $\mathbf{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z\rangle$.

(i) Evaluate div $\mathbf{F}$.

[5 *marks*]

(ii) Let S be the sphere of radius 1 centered at the origin. Compute the flux of $\mathbf{F}$ over S with outward pointing normal.

[6 *marks*]

(iii) Let S' be the sphere of radius 5 centered at $(0, 1, 0)$. Compute the flux of $\mathbf{F}$ over S' with outward pointing normal.

[6 *marks*]

Question 6

Let

$$\mathbf{G}(x, y, z) = \left\langle \frac{-y}{x^2 + 4y^2},\ \frac{x}{x^2 + 4y^2},\ 0 \right\rangle.$$

Prove or disprove that there is a vector field $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ with the following properties:

(i) $M(x, y, z)$, $N(x, y, z)$ and $P(x, y, z)$ have continuous partial derivatives for all $(x, y, z) \neq (0, 0, 0)$;

(ii) curl $\mathbf{F} = \mathbf{0}$ for all $(x, y, z) \neq (0, 0, 0)$;

(iii) $\mathbf{F}(x, y, 0) = \mathbf{G}(x, y, 0)$.

[10 *marks*]

END OF PAPER