NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION (2009–2010)

MA1102R Calculus

April 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **ONE (1)** section. It contains a total of **NINE (9)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [12 marks]

Evaluate the following limits.

(a) $\lim_{x \to 4} \dfrac{\sqrt{1+2x} - 3}{\sqrt{x} - 2}$.

(b) $\lim_{x \to \infty} \dfrac{1}{x} \sin\left(\dfrac{1}{x}\right)$.

Question 2 [8 marks]

Define

$$f(x) = \frac{2}{e^x - e^{-x}}, \quad x \in \mathbb{R} \backslash \{0\}.$$

(i) Find its inverse function $f^{-1}(x)$ explicitly.

(ii) Find $\dfrac{d}{dx}[f^{-1}(x)]$.

Question 3 [12 marks]

(a) Let n be a positive integer, and $a_1, \ldots, a_n \in \mathbb{R}^+$. Show that

$$\lim_{x \to 0} \left(\frac{a_1^x + \cdots + a_n^x}{n} \right)^{\frac{1}{x}} = \sqrt[n]{a_1 \cdots a_n}.$$

(b) Using the Riemann sum or otherwise, show that

$$\lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \ln 2.$$

Question 4 [12 marks]

Find the following integrals.

(a) $\displaystyle\int \frac{\sqrt{x^2 - 1}}{x^3}\, dx$.

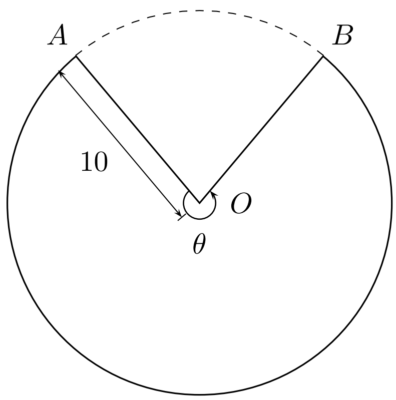
(b) $\displaystyle\int \frac{2}{(x-1)(x^2+1)}\, dx$.

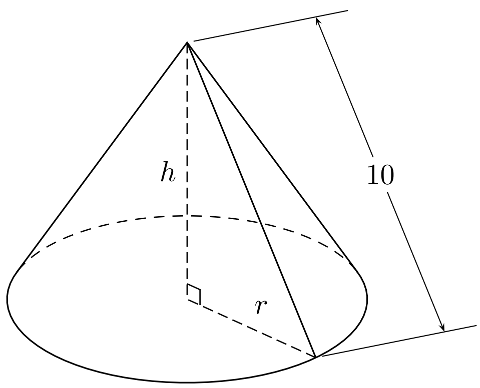
Question 5 [10 marks]

A sector AOB is to be cut out from a piece of thin cardboard in the shape of a circle of radius 10 cm. The radii OA and OB are joined together so that the sector forms the curved surface of a cone, as shown in the diagram.

(i) Denote the angle of the sector AOB by θ. Let r and h be the radius and the height of the cone respectively. Express r and h in terms of θ.

(ii) Find the angle θ such that the resulting cone has the largest volume.





Question 6 [14 marks]

A ornamental light bulb is designed by revolving the graph of

$$y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3},$$

about the x-axis, where x and y are measured in feet.

(i) Find the volume of the bulb.

(ii) Find the surface area of the bulb.

Question 7 [8 marks]

Let f be a function defined on the interval $(-1, 1)$ such that for all $x, y \in (-1, 1)$,

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}.$$

Suppose that f is differentiable at $x = 0$.

(i) Show that f is differentiable on $(-1, 1)$.

(ii) If $f'(0) = \pi/2$, find the explicit expression of $f(x)$.

Question 8 [14 marks]

(a) Solve the differential equation

$$x\frac{dy}{dx} + (1-x)y = e^{2x}, \quad x > 0.$$

with the initial condition $\lim_{x\to 0^+} y(x) = 1$.

(b) A house mortgage is a loan that is to be paid over a fixed period of time. Let $A(t)$ be the loan at time t. Then it is modeled by the differential equation

$$\frac{dA(t)}{dt} = rA(t) - 12P,$$

where r is the yearly continuously compounded interest, and $\$P$ is the monthly payment.

Suppose a 20-year mortgage of $1 million is borrowed at 5% interest compounded continuously per year. Evaluate the monthly payment.

Question 9 [10 marks]

Let f be a continuous function defined on $[0,\pi]$. Suppose that

$$\int_0^{\pi} f(x)\sin x\,dx = 0 \quad \text{and} \quad \int_0^{\pi} f(x)\cos x\,dx = 0.$$

Prove that $f(x) = 0$ has at least two real roots in $(0,\pi)$.

End of Paper