Student Number: | U/T/A* | – | | | – | | | | | |

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2009-2010

MA1101R LINEAR ALGEBRA I

April/May 2010 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper contains a total of **Five (5)** questions and comprises **Twenty Three (23)** printed pages.

3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.

4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
Total	

Question 1 [20 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

(i) [3 marks] Write down a basis for the row space of $\boldsymbol{A}$, and a basis for the column space of $\boldsymbol{A}$.

(ii) [4 marks] Find a basis for the nullspace of $\boldsymbol{A}$. Show your working.

(iii) [3 marks] Find rank($\boldsymbol{A}$), nullity($\boldsymbol{A}$) and nullity($\boldsymbol{A^T}$).

(iv) [3 marks] Extend the basis for the nullspace of $\boldsymbol{A}$ found in (ii) to a basis for $\mathbb{R}^5$.

(v) [3 marks] Find a non-zero vector that is contained in both the row space and the nullspace of $\boldsymbol{A}$.

(vi) [4 marks] Is it possible to find a matrix $\boldsymbol{B}$ such that $\boldsymbol{AB}$ is an invertible matrix?

Justify your answers for parts (iv) to (vi).

Show your working below and on the next three pages.

(Working spaces for Question 1 - Indicate your parts clearly)

(More working spaces for Question 1)

(More working spaces for Question 1)

Question 2 [20 marks]

(a) Let $\boldsymbol{A}$ be the matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

(i) [2 marks] Write down all the eigenvalues of $\boldsymbol{A}$.

(ii) [6 marks] Find a basis for the eigenspace of $\boldsymbol{A}$ associated with each of the eigenvalues. Show your working.

(iii) [2 marks] Is $\boldsymbol{A}$ a diagonalizable matrix? Why?

(b) Let $\boldsymbol{B}$ be a 2×2 matrix such that $\boldsymbol{B}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\boldsymbol{B}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(i) [4 marks] Write down an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$. Briefly explain how you obtain the answers.

(ii) [3 marks] Let n be a positive integer. Write $\boldsymbol{B}^n$ in the form $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ where the entries b_{ij} are in terms of n.

(iii) [3 marks] Is it possible to find a non-zero column vector $\boldsymbol{v}$ such that $\boldsymbol{B}\boldsymbol{v} = \boldsymbol{v}$? Justify your answer.

Show your working below and on the next three pages.

(Working spaces for Question 2 - Indicate your parts clearly)

(More working spaces for Question 2)

(More working spaces for Question 2)

Question 3 [20 marks]

(a) (i) [4 marks] Explain clearly why the following two sets are bases for $\mathbb{R}^3$.

- $S = \{(1,1,0), (0,1,1), (0,0,1)\}$
- $T = \{(1,0,1), (0,1,1), (0,1,0)\}$

(ii) [3 marks] Find the transition matrix from T to S. Show your working.

(iii) [4 marks] Suppose the coordinate vector of $\boldsymbol{w}$ with respect to the basis T is given by $(\boldsymbol{w})_T = (1, 2, -1)$. Find $\boldsymbol{w}$ and $(\boldsymbol{w})_S$. Show your working.

(iv) [3 marks] Is there any non-zero vector $\boldsymbol{v} \in \mathbb{R}^3$ such that $(\boldsymbol{v})_T = (\boldsymbol{v})_S$? Justify your answer.

(b) [6 marks] Let U and V be two subspaces of $\mathbb{R}^4$ such that $\dim U = 2$ and $\dim V = 3$. Determine whether each of the following statements is true or false. Justify your answers.

(i) $\dim(U \cap V) \geq 1$.

(ii) If U is not a subset of V, then $\dim(U \cap V) = 1$.

Show your working below and on the next three pages.

(Working spaces for Question 3 - Indicate your parts clearly)

(More working spaces for Question 3)

(More working spaces for Question 3)

Question 4 [20 marks]

(a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(i) [2 marks] Show that $\boldsymbol{Ax} = \boldsymbol{b}$ is an inconsistent system.

(ii) [5 marks] Find a least squares solution of $\boldsymbol{Ax} = \boldsymbol{b}$. Show your working.

(iii) [2 marks] Find the projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

(iv) [5 marks] Extend the set $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ to an orthogonal basis for $\mathbb{R}^3$ and normalize the vectors to get an <u>orthonormal</u> basis S for $\mathbb{R}^3$. Show your working.

(v) [2 marks] Find the coordinate vector $(\boldsymbol{b})_S$ of the vector $\boldsymbol{b}$ with respect to the <u>orthonormal</u> basis S in (iv).

(b) [4 marks] For an $m \times n$ matrix $\boldsymbol{A}$ and $m \times 1$ matrix $\boldsymbol{b}$, let $\boldsymbol{p}$ be the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$. Show that $\boldsymbol{b} - \boldsymbol{p}$ is a solution of $\boldsymbol{A^T x} = \boldsymbol{0}$.

Show your working below and on the next three pages.

(Working spaces for Question 4 - Indicate your parts clearly)

(More working spaces for Question 4)

(More working spaces for Question 4)

Question 5 [20 marks]

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix},\ T\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix},\ T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

(i) [2 marks] Find $T\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\\0\end{pmatrix}$. Show your working.

(ii) [2 marks] Write down the standard matrix for T.

(iii) [4 marks] Write down the kernel of T as a linear span. Show your working.

(iv) [3 marks] Is it true that every vector in $\mathbb{R}^2$ is an image under T? Justify your answer.

(v) [2 marks] Suppose $S : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with standard matrix $\begin{pmatrix}1 & 1\\1 & 1\\1 & 1\end{pmatrix}$. Write down the formula for the composition $S \circ T$.

(vi) [3 marks] Find all the vectors $\boldsymbol{v}$ in $\mathbb{R}^3$ such that $(S \circ T)(\boldsymbol{v}) = \boldsymbol{v}$, where S is the linear transformation in (v). Show your working.

(vii) [4 marks] Find the equation of the plane in $\mathbb{R}^3$ that is transformed to the line $x - y = 0$ in $\mathbb{R}^2$ under T. Show your working.

Show your working below and on the next three pages.

(Working spaces for Question 5 - Indicate your parts clearly)

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