NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA5213 ADVANCED PARTIAL DIFFERENTIAL EQUATIONS

December 2009 — Time allowed : 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages including this page.

2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

Let $u \in C^1_{loc}(\mathbf{R}^n)$ and assume it satisfies the following condition:

$$\int_{\mathbf{R}^n} \sum_{i,j=1}^{n} a^{ij} D_i u D_j \phi \, dx = 0 \quad \text{for all } \phi \in H^1_0(\mathbf{R}^n), \tag{1}$$

where a^{ij} are constants, $a^{ij} = a^{ji}$ for all $1 \le i, j \le n$, and there exist constants $0 < \lambda < \Lambda$ such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j \le \Lambda |\xi|^2 \quad \text{for all } x \in \mathbf{R}^n, \ \xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n.$$

(a) Prove that u is a C^∞ function.

(b) Assume u is also bounded in $\mathbf{R}^n$. Prove that u is constant on $\mathbf{R}^n$.

(c) Assume now a^{ij} are continuous functions for all $1 \le i, j \le n$. Is the conclusion (b) still true? Prove your answer.

Question 2 [15 marks]

Let Ω be an open subset of $\mathbf{R}^n$ with boundary portion $T \subseteq \{x \in \mathbf{R}^n : x_n = 0\}$, and let (a^{ij}) be an $n \times n$ constant matrix which is positive-definite, that is, there exist positive constants λ and Λ, $0 < \lambda < \Lambda$, such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j \le \Lambda |\xi|^2, \quad \xi \in \mathbf{R}^n.$$

Assume $u \in C^2(\Omega) \cap C(\Omega \cup T)$ and it solves the following problem

$$\sum_{i,j=1}^{n} a^{ij} D_{ij} u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } T, \tag{2}$$

where $f \in C^\alpha(\Omega \cup T)$ and $0 < \alpha < 1$. Use the $C^{2+\alpha}$ estimates of harmonic functions to prove that there exists a constant $C = C(n, \alpha, \lambda, \Lambda)$ such that

$$|u|^*_{2,\alpha;\Omega \cup T} \le C\big(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega \cup T}\big).$$

Question 3 [20 marks]

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary. Consider the following boundary value problem

$$Lu = f \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega, \tag{3}$$

where the operator L is defined by

$$Lu \equiv \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u + c(x) u,$$

with $a^{ij} = a^{ji}$ for all $1 \le i, j \le n$. Assume there exist constants $0 < \lambda < \Lambda$ such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \le \Lambda |\xi|^2 \quad \text{for all } x \in \bar{\Omega},\ \xi \in \mathbf{R}^n.$$

(i) State the Schauder estimate for the classical solution of the boundary value problem (3). More precisely, state the conditions on the coefficients $a^{ij}(x)$, $b^i(x)$ and $c(x)$, on f and φ, and state the conclusion.

(ii) Give a **short outline** of the proof of the Schauder estimate.

Question 4 [20 marks]

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary. Consider the following boundary value problem

$$Lu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4}$$

where the operator L is defined by

$$Lu \equiv \sum_{i,j=1}^{n} D_j \Big(a^{ij}(x) D_i u \Big) + \sum_{i=1}^{n} b^i(x) D_i u + c(x) u,$$

with $a^{ij} = a^{ji}$ for all $1 \le i, j \le n$. Assume there exist constants $0 < \lambda < \Lambda$ such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \le \Lambda |\xi|^2 \quad \text{for all } x \in \bar{\Omega},\ \xi \in \mathbf{R}^n.$$

(i) State the H^2 estimate for the weak solution of the boundary value problem (4). More precisely, state the conditions on the coefficients $a^{ij}(x)$, $b^i(x)$ and $c(x)$ and on f, and state the conclusion.

(ii) Give a **short outline** of the proof of the H^2 estimate.

Question 5 [25 marks]

Let Ω be a bounded domain in $\mathbf{R}^n$ with $n \geq 3$, and $u \in H^1(\Omega)$ be a weak subsolution of the equation

$$\sum_{i,j=1}^{n} D_j\Big(a^{ij}(x)D_i u\Big) + \sum_{i=1}^{n} b^i(x)D_i u + c(x)u = \operatorname{div}\,\mathbf{g}(x),$$

where $a^{ij}(x)$, $b^i(x)$ and $c(x)$ are measurable functions on $\bar{\Omega}$, $a^{ij}(x) = a^{ji}(x)$ for all $1 \leq i, j \leq n$. Assume there exist positive constants λ, Λ and μ such that, for all $x \in \Omega$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$,

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad \sum_{i=1}^{n} \frac{|b^i(x)|^2}{\lambda^2} + \frac{|c(x)|}{\lambda} \leq \mu^2,$$

and

$$\mathbf{g} \in L^q(\Omega, \mathbf{R}^n), \quad \|\mathbf{g}\|_{L^q(\Omega)} > 0$$

for some $q > n$.

Assume also u is a **bounded** function in $\bar{\Omega}$ and $u \leq 0$ on $\partial\Omega$. Use the DiGiorgi-Moser iteration method to show that there exists a constant $C = C(n, q, \mu, |\Omega|) > 0$ such that

$$\sup_{\Omega} u \leq C\big(\|u^+\|_{L^2(\Omega)} + \lambda^{-1}\|\mathbf{g}\|_{L^q(\Omega)}\big).$$

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