NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA5205 Graduate Analysis I

November 30, 2009 — Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper comprises **THREE (3)** printed pages.
2. This paper consists of **FIVE (5)** questions. Answer **ALL** of them.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Answer **all** *the questions in this paper*

Question 1 [20 marks]

Let F be a closed subset of R^1 and let $\delta(x) = \delta(x, F)$ be the corresponding distance function of F. Suppose $\lambda > 0$ is a constant and f is a non-negative integrable function over R^1. Show that the function

$$\int_{R^1} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}} dy$$

is integrable over F and so it is finite almost everywhere in F.

Question 2 [20 marks]

Let Ω be a subset of R^n such that $|\Omega| = 1$. Suppose $h : \Omega \to [0, \infty)$ is a non-negative measurable function. Define $\alpha = \int_\Omega h$. Show that

$$\sqrt{1+\alpha^2} \leq \int_\Omega \sqrt{1+h^2} \leq 1+\alpha.$$

Question 3 [20 marks]

Define $f(x) = x^3$ if x is a rational number in the interval $[0, 1]$ and $f(x) = x^5$ if x is an irrational number in the same interval. For a positive constant $\alpha > 0$, find the limit

$$\lim_{n\to\infty} \int_0^1 n \ln(1 + [\frac{f(x)}{n}]^\alpha) dx.$$

Question 4 [20 marks]

Let $C \subset [0, 1]$ be the Cantor set. Define the function f on $[0, 1]$ as follows: $f(x) = 0$ if $x \in C$ and $f(x) = n$ if x is in one of the sub-intervals deleted in $n-$stage. Calculate the integral $\int_{[0,1]} f$.

Question 5 [20 marks]

Suppose $g > 0$ on R^n is in $L^1(R^n)$. Let $f \in L^\infty(R^n)$ such that $\|f\|_{L^\infty} > 0$. Define

$$\alpha_k = \int_{R^n} |f|^k g, \quad (k = 1, 2, 3, \cdots).$$

Show that

$$\lim_{k\to\infty} \frac{\alpha_{k+1}}{\alpha_k} = \|f\|_{L^\infty}.$$

END OF PAPER