NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA5203 Graduate Algebra I

November 2009 — Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **THREE (3)** questions and comprises **THREE (3)** printed pages.

2. Answer **ALL** questions. Marks for each question may not be the same; they are indicated in the beginning of the question.

Answer ***all*** *questions.*

Question 1 [30 marks]

Let G be a finite group.

(a) Let p be the least prime divisor of G. Prove that any subgroup of G of index p is normal in G.

(b) Let q be a prime divisor of G, and suppose that whenever Q_1 and Q_2 are two distinct Sylow q-subgroups of G, $Q_1 \cap Q_2$ is a subgroup of Q_1 of index at least q^a. Show that the number n_q of Sylow q-subgroups of G satisfies $n_q \equiv 1 \pmod{q^a}$.

(c) Suppose that $|G| = 432$ $(= 2^4 \cdot 3^3)$. Show that G is not simple.
(You may NOT assume Burnside's $p^a q^b$ Theorem without proof.)

Question 2 [40 marks]

Let p be a prime integer, and let $n \in \mathbb{Z}^+$. Recall that an element ζ in a field extension of $\mathbb{F}_p$ is a primitive n-root of unity if and only if $\zeta^n = 1$ and $\zeta^i \neq 1$ for all $1 \leq i < n$.

(a) Show that if p divides n in $\mathbb{Z}$, then primitive n-root of unity does not exist in any field extension of $\mathbb{F}_p$.

(b) Suppose that p does not divide n in $\mathbb{Z}$, and let k be the order of p in $\mathbb{Z}_n^*$. Show that $\mathbb{F}_{p^m}$ has a primitive n-th root of unity if and only if k divides m (in $\mathbb{Z}$).

(c) Suppose further that p is odd.

(i) Show that a primitive 8-th root of unity ζ exists in some field extension of $\mathbb{F}_p$.

(ii) Let $\alpha = \zeta + \zeta^{-1}$. Show that

(A) $\alpha^2 = 2$;

(B) $\alpha^p = \alpha$ if and only if $p \equiv \pm 1 \pmod 8$. (You may NOT assume any result pertaining to quadratic residues without proof.)

(iii) Deduce, or otherwise, that there exists $\beta \in \mathbb{F}_p$ such that $\beta^2 = 2$ if and only if $p \equiv \pm 1 \pmod 8$.

Question 3 [30 marks]

Let A be a commutative ring with 1, and let $A[[x]]$ be the ring of formal power series in x with coefficients in A, i.e.

$$A[[x]] = \{\sum_{i=0}^{\infty} a_i x^i \mid a_0, a_1, \ldots \in A\}.$$

Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in A[[x]]$.

(a) Show that $f(x)$ is a unit in $A[[x]]$ if and only if a_0 is a unit of A.

(b) Show that $f(x) \in J(A[[x]])$ if and only if $a_0 \in J(A)$.
(For a commutative ring R with 1, $J(R)$ denotes the Jacobson radical of R.)

(c) Show that if $f(x)$ is nilpotent, then a_i is nilpotent for all $i \in \mathbb{Z}_{\geq 0}$.

(d) Show that if A is Noetherian, and a_i is nilpotent for all $i \in \mathbb{Z}_{\geq 0}$, then $f(x)$ is nilpotent.

(e) Show that if A is Noetherian, then $A[[x]]$ is Noetherian.
(You may assume Hilbert Basis Theorem without proof if necessary.)

END OF PAPER