NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA4247 Complex Analysis II

December 2009 – Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE** (5) questions and comprises **FOUR** (4) printed pages.
2. Answer **ALL** questions in the examination paper. Each question carries 20 marks.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.
4. Candidates are allowed to bring in **TWO** (2) double-sided A4-sized handwritten helpsheets.

Convention: *Throughout this paper,* $\mathbb{C}$ *denotes the set of complex numbers, and* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$*. For* $a \in \mathbb{C}$ *and* $r > 0$*,* $B(a,r)$ *denotes the open ball* $\{z \in \mathbb{C} : |z-a| < r\}$ *and* $\overline{B(a,r)}$ *denotes the closed ball* $\{z \in \mathbb{C} : |z-a| \leq r\}$*. For* $z \in \mathbb{C}$*,* $Re\, z$ *denotes the real part of* z*,* $Im\, z$ *denotes the imaginary part of* z*, and* $\bar{z}$ *denotes the complex conjugate of* z*. Futhermore, for* $z = x + iy$*,* $x = Re\, z$ *and* $y = Im\, z$ *unless otherwise stated.*

1. [20 marks] Determine if each of the following statements is true or false. Justify your answers carefully.

(a) There exists an analytic function f on $B(0,1)$ such that $|f(z)| \leq 4$ for all $z \in B(0,1)$, $f(0) = 0$, and $f(\frac{1}{2}) = \pi i$.

(b) Suppose $f(z)$ and $g(z)$ are *meromorphic* functions on $\mathbb{C}$ such that $f(z) = g(z)$ for infinitely many $z \in B(0,1)$. Then $f(z) = g(z)$ for all $z \in \mathbb{C}$ where f and g are non-singular.

(c) The exponential function $f(z) = e^z$ is *never* one-to-one on any open ball $B(a,r)$ with $r > 4$.

(d) There exists an analytic isomorphism from the punctured domain

$$D = \{z = x + iy \in \mathbb{C} \ : \ |x| < 2, \quad |y| < 2, \quad z \neq 1 + i\}$$

to the punctured unit ball

$$B'(0,1) := B(0,1) \setminus \{0\} = \{z \in \mathbb{C} \ : \ 0 < |z| < 1\}.$$

(e) There exists a harmonic function $u(x,y)$ on the closed ball $\overline{B(0,1)}$ such that $u(0,0) = 1/2$ and $u(\cos t, \sin t) = \cos t$, for $t \in [0, 2\pi]$.

2. [20 marks]

(a) Let $f(z) = z^6 + 3z + 1$. Find the maximum and minimum of $|f(z)|$ in the closed unit ball $\overline{B(0,1)}$.

(b) Describe geometrically the set of all points $z \in \mathbb{C}$ such that the following equation holds for the cross ratio:

$$(z, \bar{z}; 0, 1) = -1.$$

(c) Find the values of the real constants a and b such that

$$u(x, y) = x^2 + ay^2 - e^x \sin(by)$$

is a harmonic function on $\mathbb{R}^2$, and find the harmonic conjugate $v(x, y)$ of u such that $v(0, 0) = 3$.

3. [20 marks]

(a) Evaluate

$$\frac{1}{2\pi i} \int_{|z|=2} \frac{f'(z)}{f(z)} dz,$$

where $f(z) = \dfrac{\sin z \cos z}{z^7 - z^5 + z^3 - z}$, and the circle $|z| = 2$ is positively oriented.

(b) Determine all entire functions $f(z)$ such that $\operatorname{Im} f(z) \neq 0$ for all $z \in \mathbb{C}$ and $f(0) = i$.

(c) Let

$$f_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}.$$

Prove that for any $R > 0$, there exists a positive integer N such that for all $n > N$, $f_n(z)$ has no zeroes in $B(0, R)$ (you should state clearly any standard results used in your proof).

4. [20 marks]

(a) Find an analytic isomorphism from the cut plane $\mathbb{C} \setminus [0, \infty)$ to the unit ball $B(0,1)$ which maps -2 to 0. You may leave your answer as a composition of mappings.

(b) Find a Möbius transformation f that maps the upper half plane

$$H := \{z \in \mathbb{C} \ : \ \operatorname{Im} z > 0\}$$

bijectively to the ball $B(0,2)$ such that $f(i) = 1$ and $f(\infty) = -2$. Write your answer in the form $f(z) = \frac{az+b}{cz+d}$.

(c) Suppose that C_1 and C_2 are two circles that intersect transversely (that is, not tangentially) at two points. Find all pairs of points z and z^* (where z and z^* are not necessarily distinct) which are symmetric with respect to both C_1 and C_2. Justify your answer.

5. [20 marks]

(a) Find the largest possible r such that the function $f(z) = z^2 + 2z$ is one-to-one on $B(0,r)$. Justify your answer carefully.

(b) Suppose that $f(z)$ is analytic in the domain Ω and $|(f(z))^2 - 1| < 1$ for all $z \in \Omega$. Show that either $\operatorname{Re} f(z) > 0$ for all $z \in \Omega$ or $\operatorname{Re} f(z) < 0$ for all $z \in \Omega$.

(c) Suppose that $f(z)$ is analytic on $B(0,1)$ such that $|f(z)| \leq 1$ for all $z \in B(0,1)$ and f has a zero of order 3 at $z = 0$, that is, $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) \neq 0$. Prove that

$$|f'''(0)| \leq 6.$$

If $|f'''(0)| = 6$, determine the possible forms that $f(z)$ can take.

END OF PAPER