NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER EXAMINATION FOR THE DEGREE OF B.SC.

SEMESTER 1 EXAMINATION 2009–2010

MA4230 Matrix Computation

November 2009– Time allowed : 2.5 hours

Instructions to Candidates

1. This examination paper contains a total of **Five (5)** questions and comprises **Three (3)** printed pages.
2. Answer **ALL** questions.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.
4. All questions carry equal marks.

Question 1 [20 marks]

Let $A = \begin{bmatrix} 10 & 1 & 0 \\ 3 & 10 & 4 \\ 4 & 1 & 3 \end{bmatrix}$. Compute an orthogonal matrix $Q \in \mathbf{R}^{3\times 3}$ and a matrix $R \in \mathbf{R}^{3\times 3}$ of the form

$$R = \begin{bmatrix} 0 & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \quad r_{21} < 0,\ r_{12} < 0,\ r_{33} < 0,$$

such that $A = QR$.

Question 2 [20 marks]

(i) Let $W = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \in \mathbf{R}^{n\times n}$ be a given orthogonal matrix. Find an eigenvalue decomposition for the matrix

$$H = I - 2w_n w_n^T.$$

(ii) Let $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$. Show, without computing the eigenvalues, that $|\lambda| < 4$ for each eigenvalue λ of A. Apply three iterations of the Power Method with $v^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ to estimate an eigenvalue of A.

Question 3 [20 marks]

Let

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 4 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 7 \\ 9 \\ 10 \end{bmatrix}.$$

Find the reduced QR factorization of A and solve the following linear least squares problem

$$\min_{z \in \text{Range}(A)} \|b - z\|_2.$$

Question 4 [20 marks]

(i) Let $A, B \in \mathbf{R}^{3\times 3}$. Assume that $B^T A$ can be decomposed into

$$B^T A = U\Sigma V^T,$$

where $U, V \in \mathbf{R}^{3\times 3}$ are two known orthogonal matrices, and $\Sigma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$. Find a matrix $Q \in \mathbf{R}^{3\times 3}$ such that $QQ^T = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$ and

$$\|A - BQ\|_F \leq \|A - BX\|_F$$

for all matrices $X \in \mathbf{R}^{3\times 3}$ satisfying $XX^T = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$.

(ii) Let $A \in \mathbf{R}^{100\times 100}$. Assume that $\text{rank}(A) = 12$ and

$$\min\{\|A - B\|_2 \mid B \in \mathbf{R}^{100\times 100},\ \ \text{rank}(B) \leq 11\} = 1.52.$$

Find the smallest non-zero eigenvalue of $A^T A$.

(iii) Let $B \in \mathbf{R}^{8\times 8}$ with $\|B\|_2 = 0.5$. Let

$$A = \begin{bmatrix} I & B \\ B^T & I \end{bmatrix}.$$

Compute $\|A\|_2 \|A^{-1}\|_2 - 3$.

Question 5 [20 marks]

(i) Let $A \in \mathbf{R}^{m\times n}$. Assume that $m \geq n$ and the SVD of A^T is known. Find $Q \in \mathbf{R}^{m\times n}$ with $Q^T Q = I$ and a symmetric and positive semi-definite matrix $P \in \mathbf{R}^{n\times n}$ such that $A = QP$.

(ii) Let $q \in \mathbf{R}^n$ be a unit vector and $d \in \mathbf{R}^n$ be any vector orthogonal to q. Compute

$$\|(q + d)q^T - I\|_2 - \|q + d\|_2.$$

— **END OF PAPER** —