NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2009-2010

MA3227 Numerical Analysis II

November 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **SIX (6)** printed pages.

2. Answer **ALL** questions in **Section A**. Each question in Section A carries 15 marks.

3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

4. This is a closed book exam. However, candidates are allowed to bring an A4 sized help sheet which can be written on both sides.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **ALL** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [15 marks]

Consider solving $Ax = b$ by iterative method, where $A \in \mathbb{R}^{n\times n}$ and $x, b \in \mathbb{R}^n$. If there is a matrix P that is *close to* A and is *easy to invert*, then we can write $Ax = b$ as

$$Px = (P - A)x + b$$

and obtain an iterative method

$$Px^{(k)} = (P - A)x^{(k-1)} + b.$$

(a) Write down the P's associated with Jacobi iteration and Gauss-Seidel iteration.

(b) Prove that if $\rho(I - P^{-1}A) < 1$ (where $\rho(I - P^{-1}A)$ is the spectral radius of $I - P^{-1}A$), then for any initial vector $x^{(0)}$, $\|x^{(k)} - x\| \to 0$ for some vector norm $\|\cdot\|$, where x is the exact solution of $Ax = b$.

Question 2 [15 marks]

Consider the ODE $y'(t) = \lambda y(t)$ with initial condition $y(0) = y_0$, where $\lambda < 0$ is a constant.

(a) If you use backward Euler method, prove that $|y_n| \leq |y_0|$ for any positive step size Δt.

(b) If you use forward Euler method, prove that if $0 \leq \Delta t \leq \frac{2}{-\lambda}$, then $|y_n| \leq |y_0|$.

Question 3 [15 marks]

Consider the ODE $y'(t) = f(t, y(t))$ with initial condition $y(0) = y_0$. If we want to solve it by forward Euler method, we should set

$$y_{n+1} = y_n + \Delta t f(t_n, y_n), \tag{1}$$

where $t_n = n\Delta t$. Denote the exact solution by $y(t)$ and let

$$e_n = y(t_n) - y_n$$

be the error at t_n. Note that $e_0 = 0$. Assume there are positive constants T, a and L such that for any $t \in [0, T]$, any $z_i \in \mathbb{R}$ $(i = 1, 2)$ and any $\Delta t \in [0, a]$,

$$|f(t, z_1) - f(t, z_2)| \leq L|z_1 - z_2|.$$

(a) Let

$$\tau_n = y(t_{n+1}) - y(t_n) - \Delta t f(t_n, y(t_n)).$$

be the local truncation error. Prove that $|\tau_n| \leq C\Delta t^2$ where C is a constant that is independent of Δt. You may assume that $y(t) \in C^2([0, T]) = \{$ functions whose derivatives up to 2nd order are continuous on $[0, T]$ $\}$.

(b) Prove that

$$|e_n| \leq C\frac{e^{LT} - 1}{L}\Delta t, \qquad \text{whenever } 0 \leq n \leq T/\Delta t.$$

Question 4 [15 marks]

(a) For any orthonormal matrix $H \in \mathbb{R}^{n\times n}$ and any column vector $\boldsymbol{w} \in \mathbb{R}^n$, prove that $\|H\boldsymbol{w}\|_2 = \|\boldsymbol{w}\|_2$.

(b) Determine the Householder matrix H so that $H\boldsymbol{w} = \|\boldsymbol{w}\|_2\boldsymbol{e}_1$, where $\boldsymbol{e}_1$ is the transpose of $(1, 0, ..., 0) \in \mathbb{R}^n$.

SECTION B

Answer not more than **TWO** *questions in this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

The following are three different methods for solving $y'(t) = f(t, y(t))$:

(A) $y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(t_n, y_n) + f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n))\right]$,

(B) $y_{n+1} = y_n + \Delta t f\left(t_n + \frac{\Delta t}{2}, y_n + \frac{\Delta t}{2} f(t_n, y_n)\right)$,

(C) $y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$.

Consider the problem

$$y' = i\lambda y, \qquad y(0) = y_0, \tag{2}$$

where $i = \sqrt{-1}$ and $\lambda \in \mathbb{R}$.

(a) Show that you will have the same recursive relationship between y_n and y_{n+1} when you apply either method (A) or method (B) to solve (2).

(b) When method (A) or method (B) is applied to solve (2), prove that for any fixed Δt, $|y_n| \to \infty$ as $n \to \infty$.

(c) Prove that if you apply method (C) to solve (2), you will always have $|y_n| = |y_0|$.

Question 6 [20 marks]

Recall the QR method (In our case, Q is an orthonormal matrix and R is an upper triangular matrix):

$A^{(0)} = A$
for $k = 1, 2, ...$
  $Q^{(k)} R^{(k)} = A^{(k-1)}$
  $A^{(k)} = R^{(k)} Q^{(k)}$
end

(a) Show that $A^{(k)}$ and $A^{(k-1)}$ have the same eigenvalues.

(b) Consider the matrix $A = \boldsymbol{u}\boldsymbol{v}^{\tau}$ where $\boldsymbol{u},\ \boldsymbol{v} \in \mathbb{R}^n$ are column vectors and $\boldsymbol{v}^{\tau}$ is the transpose of $\boldsymbol{v}$.

(1) Show that there **is** a QR decomposition of A. [Hint: Comparing both sides of the following:

$$QR = (q_1, ..., q_n)R = \boldsymbol{u}\boldsymbol{v}^{\tau}.$$

You are only asked to find one such QR decomposition.]

(2) Let $A = QR$ be a QR decomposition of the matrix $A = \boldsymbol{u}\boldsymbol{v}^{\tau}$. Show that the matrix RQ is already an upper triangular matrix and find all the eigenvalues of A.

Question 7 [20 marks]
Recall that the r-th order Adams-Bashforth scheme for $y'(t) = f(t, y(t))$, $y(0) = y_0$ has the following form

$$\frac{y_{n+1} - y_n}{\Delta t} = a_1 f_n + a_2 f_{n-1} + \cdots + a_r f_{n+1-r}, \tag{3}$$

where $f_k = f(t_k, y_k)$ for $k = n, ..., n + 1 - r$ and $t_k = n\Delta t$.

(a) Derive the 2nd order Adams-Bashforth scheme. Show all the necessary steps.

(b) The following is part of a Matlab code by N. L. Trefethen which plots the boundaries of stability regions of the 1st to the 3rd order Adams-Bashforth schemes:

```
% stability regions for Adams-Bashforth:
  z = exp(1i*pi*(0:200)/100); r = z-1;
  s = 1; plot(r./s)                                % order 1
  s = (3-1./z)/2; plot(r./s)                       % order 2
  s = (23-16./z+5./z.^2)/12; plot(r./s)            % order 3
  axis([-2.5 .5 -1.5 1.5]), axis square, grid on
```

You may ignore the last line if you are not familiar with those graphing functions. Show how the above code is related to the stability regions of the 2nd order Adams-Bashforth schemes. [Hint: Recall that the stability region for a numerical method of ODE is a subset of the complex plane consisting of those $\lambda\Delta t$ for which the numerical method produces bounded solutions when applied to the scalar linear model problem $y'(t) = \lambda y(t)$ $(\lambda \in \mathbb{C})$ with step size Δt.]

END OF PAPER