NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2009-2010

MA2101 Linear Algebra II

December 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring one piece of A4-sized two-sided help sheet into the examination room.

2. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **EIGHT (8)** questions and comprises **SIX (6)** printed pages.

3. Answer **ALL** questions in **Section A**. Section A carries a total of 70 marks.

4. Answer not more than **TWO (2)** questions from **Section B**. Section B carries a total of 30 marks.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 70 marks.*

Question 1.

Let

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad \mathcal{B}_2 = \{1, x, x^2\}$$

and let $T : \mathrm{M}_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + 3c + d) + (3b - 2a - 2d)x - (b + 2c)x^2.$$

(i) Find the matrix $[T]_{\mathcal{B}_2,\mathcal{B}_1}$. [5 marks]

(ii) Find a basis for $\mathcal{R}(T)$. [6 marks]

(iii) Find the nullity of T. [4 marks]

(iv) Let $S : P_2(\mathbb{R}) \to \mathrm{M}_2(\mathbb{R})$ be the linear transformation such that

$$[S]_{\mathcal{B}_1,\mathcal{B}_2} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}.$$

Find a formula for $(T \circ S)(a + bx + cx^2)$. [5 marks]

Question 2.

(a) Let A be a complex square matrix such that its characteristic polynomial $c_A(x)$ is given by

$$c_A(x) = (x + 1)(x - i)^2(x - 3)^3$$

and the eigenspace of A corresponding to the eigenvalue $\lambda = 3$ has dimension 2.

(i) List all the possible Jordan canonical forms of A. Be sure that no two matrices in your list are similar to each other. [5 marks]

(ii) For each of the matrices in your list, state its minimal polynomial. [3 marks]

(b) Let B be the real 3×3 matrix

$$B = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}.$$

Find the minimal polynomial of B. [7 marks]

Question 3.

Let $M_2(\mathbb{R})$ be given the inner product

$$\langle A, B \rangle = \mathrm{Tr}(B^T A), \qquad A, B \in M_2(\mathbb{R}),$$

and let W_1 be the subspace of $M_2(\mathbb{R})$ spanned by $\{A_1, A_2\}$ where

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

(i) Find an orthonormal basis for W_1. [5 marks]

(ii) Let $F = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix}$. Find $\mathbf{proj}_{W_1}(F)$. [5 marks]

(iii) Suppose W_2 is a subspace of $M_2(\mathbb{R})$ such that $W_2 \subseteq W_1^\perp$ and

$$\mathbf{proj}_{W_1 \oplus W_2}(F) = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}.$$

Find the smallest value in the set $\{\|F - X\| : X \in W_2\}$. [5 marks]

Question 4.

For each of the following statements, determine whether it is true or false. If the statement is true, give a proof. If it is false, give a counterexample.

(a) If S_1 and S_2 are two linearly independent subsets of a vector space V such that $S_1 \cap S_2 = \emptyset$, then $S_1 \cup S_2$ is also linearly independent. [5 marks]

(b) Let $\mathbb{F}$ be a field and $A \in \mathrm{M}_n(\mathbb{F})$. Let $T : \mathrm{M}_n(\mathbb{F}) \to \mathrm{M}_n(\mathbb{F})$ be the linear operator defined by

$$T(X) = AX - XA, \quad X \in \mathrm{M}_n(\mathbb{F}).$$

Then $\det(T) = 0$. [5 marks]

Question 5.

Let $P(\mathbb{R})$ be the real vector space consisting of all polynomials with coefficients in $\mathbb{R}$, and let $T : P(\mathbb{R}) \to P(\mathbb{R})$ be the linear operator defined by

$$T[f(x)] = f'(x),$$

where $f'(x)$ is the derivative of $f(x)$. More explicitly,

$$T(a_0 + a_1 x + \cdots + a_k x^k) = a_1 + 2a_2 x + \cdots + k a_k x^{k-1}.$$

(i) Prove that for each positive integer n, the subset

$$\{T^0, T^1, T^2, ..., T^n\}$$

of $\mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ is linearly independent. Here $T^0 = I_{P(\mathbb{R})}$ is the identity operator on $P(\mathbb{R})$, $T^1 = T$ and $\mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$ denotes the real vector space of all linear operators on $P(\mathbb{R})$. [6 marks]

(ii) Let

$$S = \{T^j : \ j = 0, 1, 2, 3, ...\}.$$

Is S a basis for $\mathcal{L}(P(\mathbb{R}), P(\mathbb{R}))$? Justify your answer. [4 marks]

SECTION B

Answer not more than **two** *questions from this section. Section B carries a total of 30 marks.*

Question 6.

Let V be a vector space over a field $\mathbb{F}$, and let U_1 and U_2 be subspaces of V. Suppose that $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ is a basis for U_1 and $\{\mathbf{v}_1, ..., \mathbf{v}_m\}$ is a basis for U_2. Let

$$W = \{(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m) \in \mathbb{F}^{n+m} : \ \alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n + \beta_1\mathbf{v}_1 + \cdots + \beta_m\mathbf{v}_m = \mathbf{0}\}.$$

(i) Prove that W is a subspace of $\mathbb{F}^{n+m}$. [5 marks]

(ii) Prove that $\dim(W) = \dim(U_1 \cap U_2)$. [10 marks]

Question 7.

Let V be a finite dimensional vector space over the field $\mathbb{F}$, and let $I_V, T_0 : V \to V$ be defined by

$$I_V(\mathbf{v}) = \mathbf{v}, \quad T_0(\mathbf{v}) = \mathbf{0} \qquad (\mathbf{v} \in V).$$

Suppose that $T_1, ..., T_k$ are linear operators on V such that

(i) $I_V = T_1 + \cdots + T_k$,

(ii) $T_iT_j = T_0$ for all $1 \le i, j \le k$ such that $i \ne j$, and

(iii) $T_i^2 = T_i$ for all $1 \le i \le k$.

For each $1 \le i \le k$, let $\mathcal{R}(T_i)$ be the range of T_i.

(a) Prove that

$$V = \mathcal{R}(T_1) \oplus \mathcal{R}(T_2) \oplus \cdots \oplus \mathcal{R}(T_k).$$

[5 marks]

(b) Prove that if $\lambda_1, ..., \lambda_k$ are distinct elements of $\mathbb{F}$ and

$$T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k,$$

then T is diagonalizable. [10 marks]

Question 8.

Let V be a finite dimensional complex inner product space and $T : V \to V$ a self-adjoint linear operator on V.

(i) Prove that the operator $I_V + iT$ is invertible. Here I_V is the identity operator on V. [5 marks]

(ii) Prove that the operator

$$S = (I_V - iT)(I_V + iT)^{-1}$$

is a unitary operator on V. [10 marks]

END OF PAPER