NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2009 - 2010

MA1507 Advanced Calculus

November/December 2009 —— Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **TWO (2)** sections and comprises **FIVE (5)** printed pages. Section A contains **SIX (6)** questions and Section B contains **TWO (2)** questions.

2. Answer **ALL** questions in Section A and Section B. The mark for each question is indicated at the beginning of the question.

3. Candidates may use calculators or computers. However, they should lay out systematically the crucial steps in the calculations to show that they have understood the mathematics.

4. This is an open book examination. Candidates are allowed to refer to any learning material.

· ~ · SECTION A · ~ ·

Section A contains **SIX (6)** questions. Answer **ALL** questions.

Question 1 (10 marks)

(a) Find the Cartesian equation and a parametric representation of the straight line passing through the points $(3, -1, 1)$ and $(1, 1, -1)$.

(b) Find a point on and a normal to the plane $2x - y + z = 3$.

Question 2 (10 marks)

(a) Find the directional derivative of $f(x, y, z) = xye^{z-1} + yze^{x-1} + zxe^{y-1}$ at the point $(1, 1, 1)$ in the direction from the point $(1, 1, 1)$ to $(2, 3, -1)$

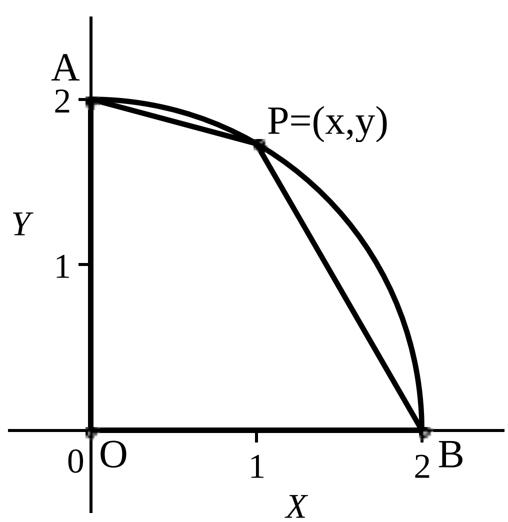
(b) If $x^2 + y^2 + z^2 - 10 = 0$, find $\left(\frac{\partial z}{\partial x}\right)_y$, $\left(\frac{\partial z}{\partial y}\right)_x$, $\frac{\partial^2 z}{\partial y \partial x}$.

Question 3 (10 marks)

The figure below shows a point $P = (x, y)$ that lies on the arc of the circle $x^2 + y^2 = 4$ in the first quadrant with end points A and B.

(i) Find the area of the quadrilateral OAPB in terms of x, y, where O is the origin.

(ii) Use the Lagrange multiplier method to find the point P on the arc that gives quadrilateral OAPB the maximum area. What is the maximum area?



Question 4 (10 marks)

Evaluate $\iint_D (x+1)\, dxdy$, where D is the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 2)$.

Question 5 (10 marks)

A thin wire in the shape of the curve $\boldsymbol{r}(t) = \left(t, t^2, t^2\right)$, $0 \leq t \leq 1$, has density t at the point $\boldsymbol{r}(t)$. Find the mass of the wire.

Question 6 (10 marks)

Evaluate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$, where $\boldsymbol{F} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + \left(z - 2x^2 - 2y^2\right)\boldsymbol{k}$ and S is the surface $x^2 + y^2 + z = 1$ in the first octant.

· ~ · ~ · END OF SECTION A · ~ · ~ ·

· ~ · SECTION B · ~ ·

Section B contains **TWO (2)** questions. Answer **ALL** questions.

Question 7 (20 marks)

(a) Consider the surfaces $z = \sqrt{a^2 + x^2 + y^2}$ and $z = \sqrt{b^2 - x^2 - y^2}$, where a, b are positive constants .

(i) Find the relationship between a and b for the two surfaces to touch one another, i.e. to contact at a point.

(ii) Find the relationship between a and b for the two surfaces to intersect at a curve.

(iii) If the two surfaces intersect, find an equation of the curve of intersection.

(iv) If S_1 is the surface $z = \sqrt{3 + x^2 + y^2}$ and S_2 is the surface $z = \sqrt{5 - x^2 - y^2}$, find the volume of the solid V bounded by the two surfaces S_1 and S_2 .

Let $\boldsymbol{F}(x, y, z) = -y\,\boldsymbol{i} + x\,\boldsymbol{j} + (x + y)z\,\boldsymbol{k}$, for $(x, y, z) \in \mathbb{R}^3$.

(v) Find $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is the intersection of the surfaces S_1 and S_2 in (a) (iv).

(vi) Find $\iint_{B_1} \boldsymbol{Curl}(\boldsymbol{F}) \cdot \boldsymbol{n}dS$ and $\iint_{B_2} \boldsymbol{Curl}(\boldsymbol{F}) \cdot \boldsymbol{n}dS$, where B_1 is the boundary surface of V below C and B_2 is boundary surface of V above C. Describe the orientations of B_1 and B_2 in your integrals.

Question 8 (20 marks)

(a) (i) Show that $\boldsymbol{F}(x, y, z) = x\left(x^2+y^2+z^2\right)^m \boldsymbol{i} + y\left(x^2+y^2+z^2\right)^m \boldsymbol{j} + z\left(x^2+y^2+z^2\right)^m \boldsymbol{k}$ is a conservative vector field on $\mathbb{R}^3$ for any nonnegative integer m.

(ii) Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is a curve with initial point $(1, 0, 0)$ and terminal point $(0, 1, 1)$.

(b) A solid pyramid P with a square base D and four triangular faces has vertices $(1, 0, 0)$, $(0, 1, 0)$, $(-1, 0, 0)$, $(0, -1, 0)$, $(0, 0, 2)$.

(i) What is the volume of the whole pyramid P?

(ii) If $\boldsymbol{F}(x, y, z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$, using divergence theorem, evaluate the surface integral $\iint_{\partial P} \boldsymbol{F} \bullet \boldsymbol{n}\, dS$, where ∂P is the boundary of P.

(iii) Evaluate $\iint_D \boldsymbol{F} \bullet \boldsymbol{n}\, dS$, where D is the base of the pyramid considered as an oriented surface with normal $\boldsymbol{n} = -\boldsymbol{k}$.

(iv) Hence, or other otherwise, find the value of the surface integral of $\boldsymbol{F}$ over each of the triangular faces of P oriented with the normal in the direction away from P.

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