NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER I EXAMINATION 2009-2010

MA1104 Multivariable Calculus

December 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring two pieces of A4-size help sheets into the examination room.

2. This examination paper contains a total of **TWELVE (12)** questions and comprises **SIX (6)** printed pages.

3. Answer **ALL** questions. The marks for the questions are not necessarily the same; marks for each question are indicated at the beginning of the question.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1. [10 marks]

Let $\mathbf{r}(t) = \langle t^4, t^3, t^2 \rangle$ denote the motion of a particle P in $\mathbb{R}^3$.

Let C denote the curve which is its path.

(i) Find the equation of the tangent line at $t = 1$.

(ii) Find the equation of the plane which contains the tangent line in (i) and the point $(2, 3, 1)$. Express you answer in the form $Ax + By + Cz = D$.

(iii) Suppose $\mathbf{s}(t) = \langle a(t), b(t), c(t) \rangle$ denotes the motion of another particle Q which travels along C such that

- $\mathbf{s}(0) = \langle 0, 0, 0 \rangle$ and
- at every point X along the curve C, the speed of Q passing through X is three times that of P.

Determine $\mathbf{s}(t) = \langle a(t), b(t), c(t) \rangle$ explicitly.

Question 2. [10 marks]

Let

$$f(x, y) = \frac{x^2 y^4}{x^4 + 2x^2 y^4 + 3y^8}.$$

(i) State the domain of $f(x, y)$. Briefly explain your answer.

(ii) Does the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

exist?
If it exists, compute its value.
If it does not exist, give a proof.

Question 3. [5 marks]

(i) Find the tangent plane of the surface

$$x^2 + 2xy^2 - 7x^3 + 3y + 1 = 0$$

at the point $P(1, 1, 1)$.

(ii) Find the equation of the line which is perpendicular to the tangent plane in (i) and passes through P.

Question 4. [5 marks]
The function

$$f(x, y) = 100 - x^2 - y^2 + 2xy$$

gives the temperature f at a point $P(x, y)$ on the xy-plane.
Let $\mathbf{u}$ denote a unit vector in $\mathbb{R}^2$.

(i) Determine all $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}}f(1, 3)$ is maximal.

(ii) Determine all $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}}f(1, 3) = 0$.

Question 5. [15 marks]

Let S denote the ellipsoid defined by

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{144} = 1.$$

(i) Find the maximal and minimal value of $f(x, y, z) = x + y + z$ on S.

(ii) Let P denote the plane defined by

$$x + y + z = -100.$$

Find the distance from the ellipsoid S to the plane P, that is, the shortest possible distance between a point A on S and a point B on the plane P.

(iii) Find a point A on S and a point B on the plane P which attain the distance computed in (ii).

Question 6. [12 marks]

Let

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Does each of the following partial derivatives (i) to (iv) below exist at the point $(0,0)$?
If it exists, compute its value.
If it does not exist, give a proof.

(i) $\dfrac{\partial f}{\partial x}(0,0)$.

(ii) $\dfrac{\partial f}{\partial y}(0,0)$.

(iii) $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$.

(iv) $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.

Hint: You may assume that for $(x,y) \neq (0,0)$,

$$\begin{aligned} \frac{\partial f}{\partial x} &= y\,\frac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2} \text{ and} \\ \frac{\partial f}{\partial y} &= x\,\frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}. \end{aligned}$$

Question 7. [5 marks]

Compute the line integral

$$\oint_C (3xy+y^2)dx + (2xy+5x^2)dy$$

where C is the closed curve $(x-1)^2+(y+2)^2=1$.

Question 8. [8 marks]

Suppose $f(x,y,z)$ is a function which satisfies

$$f(tx,ty,tz) = t^5 f(x,y,z)$$

for all $x,y,z,t \in \mathbb{R}$.

(i) Show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = Cf$$

for some constant C.

(ii) Determine the value of C in (i).

Question 9. [10 marks]

Consider the region

$$E = \{(x,y,z) : x^{2/3} + y^{2/3} + z^{2/3} \leq 1\}.$$

(i) By a change of coordinates $x = u^3, y = v^3, z = w^3$, show that the volume of E is equal to

$$\iiint_G f(u,v,w)\, dudvdw$$

where G is a region in uvw-space and $f(u,v,w)$ is a function.
Determine G and $f(u,v,w)$ explicitly.

(ii) Using (i) or otherwise, compute the volume of E.

(Hint: You may assume that

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \begin{cases} \dfrac{\pi}{2}\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} & \text{if } n \text{ is even,} \\[2ex] \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n} & \text{if } n \text{ is odd.} \end{cases}$$

Question 10. [10 marks]

Let A and B be scalars so that

$$\mathbf{F}(x, y, z) = \langle z^2 + Axy,\ x^2,\ Bxz \rangle$$

is a *conservative* vector field in $\mathbb{R}^3$.

(i) Find the values of A and B.

(ii) Find a function $f(x, y, z)$ such that $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$.

(iii) Let C be the arc $\mathbf{r}(t) = \langle 2 + t^2 + t^5, 1 + 3t - 5t^3, 3 - 3t \rangle$ for $0 \le t \le 1$.
Compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Question 11. [10 marks]

Let $\mathbf{F} = \langle 2z - x^3, \sin(yz), ze^{x+y} \rangle$. Let S denote the part of the graph of $z = 15 - 3x^2 - 5y^2$ which lies above the xy-plane.

(i) Compute $\operatorname{curl} \mathbf{F}$.

(ii) Evaluate the surface integral

$$\iint_S (\operatorname{curl} \mathbf{F} \cdot \mathbf{n}) \, d\sigma$$

where $\mathbf{n}$ points upwards.
(Hint: Consider the xy-plane.)

Question 12. [10 marks]

Let S denote the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

Let $\mathbf{n}$ denote a unit normal vector on S pointing outwards.

Let $f(x, y, z)$ be a function with second partial derivatives.

Suppose $|\nabla f|^2 = 3f$ and $\operatorname{div}(f \nabla f) = 7f$.

Compute the surface integral

$$\iint_S \nabla f \cdot \mathbf{n} \, d\sigma.$$

END OF PAPER