NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION (2009–2010)

MA1102R Calculus

November 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **ONE (1)** section. It contains a total of **NINE (9)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [10 marks]

Find the following limits.

(a) $\displaystyle\lim_{x\to\pi/4} \frac{\sec^2 x - 2\tan x}{1+\cos 4x}$.

(b) $\displaystyle\lim_{x\to 0^+} x\cos(1/\sqrt{x})$.

Question 2 [12 marks]

(a) Prove the limit $\displaystyle\lim_{x\to 2} \frac{3x^2 - x - 4}{x+1} = 2$ using the ϵ, δ-definition.

(b) Find $\dfrac{dy}{dx}$ if $y = \dfrac{(e^x+1)\sqrt{x^2+2}}{(x-8)^5}$, $x > 8$.

Question 3 [9 marks]

Consider the function $f(x) = x^3 - 9x^2 + 24x - 7$ on $\mathbb{R}$.

(i) Find the open intervals on which it is increasing and decreasing.

(ii) Find the coordinates of all its local maximum and minimum points.

(iii) Find the open intervals on which it is concave up and concave down.

(iv) Find the coordinates of all its inflection points.

Question 4 [8 marks]

A farmer needs to build a fence to enclose a rectangular region of area 1200 square meters. As one side of the region will face a main road, the farmer decides to make that side more attractive by using higher quality fencing that costs \$6 per meter. For the other three sides, he intends to use fencing that costs \$3 per meter. What dimensions of the rectangular region will minimize the cost of the fence?

Question 5 [10 marks]

Evaluate the following integrals.

(a) $\displaystyle\int \cos x \ln(\sin x)\,dx$.

(b) $\displaystyle\int_1^2 x\sqrt{2-x}\,dx$.

Question 6 [17 marks]

(a) Find the length of the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ from $x = 1$ to $x = 3$.

(b) Consider the region R bounded by $y = 2x^2$, $x = 2$ and the x-axis. For $0 < a < 2$, the vertical line $x = a$ divides R into two parts R_1 and R_2, where R_1 denotes the part on the right of $x = a$ and R_2 denotes the part on the left of $x = a$. Let V_1 be the volume of the solid generated by revolving R_1 about the x-axis, and V_2 be the volume of the solid generated by revolving R_2 about the y-axis. Find the value of a that maximizes the total volume given by $V = V_1 + V_2$.

Question 7 [13 marks]

(a) Solve the differential equation

$$x\frac{dy}{dx} + 2y = \frac{1}{x + x^3}, \quad x > 0.$$

(b) The growth of a fish population in a pond is modeled by the differential equation

$$\frac{dP}{dt} = 0.0008\,P(100 - P),$$

where $P = P(t)$ represents the size of the fish population at time t (measured in weeks). Initially the fish population has a size of 20. Derive a formula for the size of the fish population at time t.

Question 8 [10 marks]

Let f and g be functions such that f'' and g'' exist everywhere on $\mathbb{R}$. For $a < b$, suppose that $f(a) = f(b) = g(a) = g(b) = 0$, and $g''(x) \neq 0$ for every $x \in (a, b)$.

(i) Prove that $g(x) \neq 0$ for every $x \in (a, b)$.

(ii) Show that there exists a number $c \in (a, b)$ for which

$$\frac{f(c)}{g(c)} = \frac{f''(c)}{g''(c)}.$$

Question 9 [11 marks]

Let f be a continuous function on $\mathbb{R}$ such that $\lim_{x \to 0} \frac{f(x)}{x}$ exists. Define the function

$$g(x) = \int_0^1 f(xt)\, dt, \quad x \in \mathbb{R}.$$

Determine whether g' is continuous at $x = 0$. Justify your answer.

[END OF PAPER]