NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2009-2010

MA1101R Linear Algebra I

November 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FOUR (4)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.
3. Calculators may be used. However, you should lay out systematically the various steps in the calculations

Question 1 [25 marks]

(a) Dr. Ng (not his real name) works part time as a fruit seller. He goes to the fruit wholesaler every day with \$200 and buys back 3 types of fruits to sell at his stall. On a particular day, he decides to choose from oranges, grapefruits and mangoes and finds these fruits selling at the following prices:

Orange: \$0.50 per orange
Grapefruit: \$1.10 per grapefruit
Mango: \$1.50 per mango

(i) If Dr. Ng wants to buy **exactly** 100 fruits and use **all** his \$200, write down a linear system with two equations and three variables.

(ii) Determine if it is possible for Dr. Ng to make his purchases.

> **Important:** For this question, it is obvious that Dr. Ng is not allowed to buy a fraction of a fruit, neither is it possible that the number of a particular fruit he buys is negative.

(b) Find all values of k such that the linear system

$$\begin{cases} kx_1 + x_2 + x_3 = 0 \\ x_1 + 2kx_2 + x_3 = 1 \\ x_2 + x_3 = 1 \end{cases}$$

has (i) no solution; (ii) exactly one solution; (iii) infinitely many solutions.

(c) Consider the following linear system

$$\begin{cases} x + y = 1 \\ -x + y = 2 \\ 2x + y = 3 \end{cases}$$

(i) Show that the linear system is inconsistent.

(ii) Find a least squares solution to the linear system.

(iii) In general, if $\boldsymbol{y}$ is a least squares solution to a linear system $\boldsymbol{Ax} = \boldsymbol{b}$, prove that $r\boldsymbol{y}$ is a least squares solution to the linear system $\boldsymbol{Ax} = r\boldsymbol{b}$ for all non zero real numbers r.

Question 2 [25 marks]

(a) Let $\boldsymbol{A}$ be a 3×3 matrix such that

$$\boldsymbol{A} \underset{\boldsymbol{E_1}}{\xrightarrow{R_1 + 2R_2}} \underset{\boldsymbol{E_2}}{\xrightarrow{2R_3}} \underset{\boldsymbol{E_3}}{\xrightarrow{R_1 \leftrightarrow R_3}} \underset{\boldsymbol{E_4}}{\xrightarrow{R_3 - R_2}} \boldsymbol{I}.$$

(i) Write down 4 elementary matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$, $\boldsymbol{E_3}$, $\boldsymbol{E_4}$ such that

$$\boldsymbol{E_4 E_3 E_2 E_1 A} = \boldsymbol{I}.$$

(ii) If $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, solve $\boldsymbol{Ax} = \boldsymbol{b}$.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}.$$

(i) Find the standard matrix for T.

(ii) Find a basis for the range of T.

(iii) Write down $\text{rank}(T)$ and $\text{nullity}(T)$.

(c) Find a **non-zero** linear transformation $S : \mathbb{R}^4 \to \mathbb{R}^4$ such that the range of S is orthogonal to $\text{span}\{(1, 1, 1, 0), (0, -1, -1, -1)\}$.

Question 3 [25 marks]

(a) Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & 1 \end{pmatrix}.$$

Find an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^T \boldsymbol{AP}$ is a diagonal matrix.

(*Question 3 continues on next page...*)

(...continuation of Question 3.)

(b) For $n \geq 2$, let $\boldsymbol{B_n} = (b_{ij})$ be a square matrix of order n such that

$$b_{ij} = \begin{cases} 0 & \text{if } i > j \text{ or } j > i+1; \\ 1 & \text{if } j = i+1; \\ k & \text{if } i = j, \end{cases}$$

where k is a real number.

(i) Write down $\boldsymbol{B_2}$ and $\boldsymbol{B_3}$.

(ii) Find all the eigenvalues of $\boldsymbol{B_n}$.

(iii) Prove that $\boldsymbol{B_n}$ is not diagonalizable for all $n \geq 2$.

(c) Is there a symmetric matrix $\boldsymbol{C}$ of order n such that for all $x \in \mathbb{R}$, the rows of $\boldsymbol{C} - x\boldsymbol{I}$ is a basis for $\mathbb{R}^n$? Justify your answer.

Question 4 [25 marks] Let

$$\boldsymbol{u_1} = (1, 2, -1, 0) \quad \boldsymbol{u_2} = (-1, 1, 1, 3) \quad \boldsymbol{u_3} = (2, -1, 0, 1)$$
$$\boldsymbol{u_4} = (-5, -2, 3, 2) \quad \boldsymbol{u_5} = (4, 0, -2, -2) \quad \boldsymbol{u_6} = (10, -11, 0, -1).$$

(i) Show that $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is an orthogonal set.

(ii) Write each of $\boldsymbol{u_4}, \boldsymbol{u_5}, \boldsymbol{u_6}$ as linear combinations of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$.

Hint: You may wish to use the following fact:

$$\left(\begin{array}{ccc|c|c|c} 1 & -1 & 2 & -5 & 4 & 10 \\ 2 & 1 & -1 & -2 & 0 & -11 \\ -1 & 1 & 0 & 3 & -2 & 0 \\ 0 & 3 & 1 & 2 & -2 & -1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -2 & 1 & -2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

(iii) Prove that S is a basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}, \boldsymbol{u_6}\}$.

(iv) Show that $T = \{\boldsymbol{u_4}, \boldsymbol{u_5}, \boldsymbol{u_6}\}$ is also a basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}, \boldsymbol{u_6}\}$.

(v) Find the transition matrix from T to S.

(vi) Find a vector (a, b, c, d) such that the row space of the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ a & b & c & d \end{pmatrix}$$

is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}, \boldsymbol{u_6}\}$.

END OF PAPER