NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

Qualification Examination

**Analysis**

January, 2010 — Time allowed : 3 hours

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**INSTRUCTIONS TO CANDIDATES**

1. This examination paper comprises **THREE (3)** printed pages.
2. This paper consists of **EIGHT (8)** questions. Answer **ALL** of them. Marks for each question are not necessarily the same; marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

*Answer* **all** *the questions in this paper*

**Question 1** [20 marks]

Suppose $f$ is a non-negative function on $R^n$ such that $\int_{R^n} f = 1$. If $0 < p < 1$ is real number, show that

$$\int_E f^p \leq |E|^{1-p}$$

for every measurable set $E$. Similarly show that if $E$ is a measurable set with $0 < |E| < \infty$, then

$$\int_E \log f \leq -|E| \log |E|.$$

**Question 2** [15 marks]

Let $\Delta$ be the unit disk of the complex plane. Let $f$ be analytic and bounded by $M$ on $\Delta$. If $a_1, a_2, \cdots, a_n$ are among the zeros of $f$, and define

$$B(z) = \Pi_{k=1}^{n} \frac{z - a_k}{1 - \bar{a_k} z}.$$

(1) Show that $B$ is analytic on $\Delta$ and $|B(z)| = 1$ for $|z| = 1$;
(2) Show that $|f(z)| \leq M|B(z)|$ for each $z \in \Delta$.

**Question 3** [15 marks]

Show that the polynomial $P(z) = 3z^{15} + 4z^8 + 6z^5 + 19z^4 + 3z + 1$ has (i) 4 zeros for $|z| < 1$ and (ii) 11 zeros for $1 < |z| < 2$.

**Question 4** [10 marks]

Let $\{f_k\}$ be a sequence of non-negative measurable functions on a measurable set $E$ with $|E| < \infty$. Is it true that $f_k$ converges to 0 on $E$ in measure as $k \to \infty$ if and only if

$$\lim_{k \to \infty} \int_E \frac{f_k}{1 + f_k} = 0?$$

If it is true, prove it. If it is not true, provide a counterexample.

**Question 5** [10 marks]

Suppose $f(u)$ is a continuous function on $[-1,1]$. Show that

$$\int\int_{x^2+y^2+z^2=1} f(z)ds = 2\pi \int_{-1}^{1} f(z)dz.$$

**Question 6** [10 marks]

Suppose that $\sum_{n=1}^{\infty} u_n$ converges. Show that

$$\lim_{n\to\infty} \frac{u_1 + 2u_2 + 3u_3 + \cdots + nu_n}{n} = 0.$$

**Question 7** [10 marks]

Let $\phi(t)$ be a positive continuous function on $[0,\infty)$ and $f(t,x)$ be a continuous function of two variables such that $|f(t,x)| \leq \phi(t)|x|$. Suppose $\int_0^{\infty} \phi(t) < \infty$. Show that if the function $y$ satisfies the inequality

$$|y(t)| \leq \int_0^t |f(s, y(s))|ds,$$

for all $t \in [0,\infty)$, then $y(t) \equiv 0$.

**Question 8** [10 marks]

Consider the functions

$$f(x) = (\int_0^x e^{-u^2} du)^2$$

and

$$g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1} dt.$$

Show that $f(x) + g(x) = \frac{\pi}{4}$ for all $x \geq 0$ and hence $\lim_{x\to\infty} \int_0^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

**END OF PAPER**