NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

Qualification Examination

Analysis

August, 2009 — Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper comprises **THREE (3)** printed pages.
2. This paper consists of **TEN (10)** questions. Answer **ALL** of them.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Answer **all** *the questions in this paper*

Question 1 [10 marks]

Suppose f and g are both measurable function on the interval $(0,1)$ such that for all $t \in R^1$,

$$|\{x \in (0,1) : f(x) \geq t\}| = |\{x \in (0,1) : g(x) \geq t\}|.$$

Assume f and g both are monotone decreasing and continuous from left. Can you conclude that $f(x) = g(x)$ for all $x \in (0,1)$? Give the reason to support your answer.

Question 2 [10 marks]

Compute the volume of the region bounded by $(a_{11}x + a_{12}y + a_{13}z)^2 + (a_{21}x + a_{22}y + a_{23}z)^2 + (a_{31}x + a_{32}y + a_{33}z)^2 = 1$ where the determinant of the 3×3 matrix (a_{ij}) is NOT equal to zero.

Question 3 [10 marks]

Let D be a measurable set in R^n with finite measure. Suppose $\phi(x,t)$ is a real valued continuous function on $D \times R^1$ such that for almost every $x \in D$, $\phi(x,t)$ is a continuous function of t and for every real number t, $\phi(x,t)$ is measurable function of x. If $\{f_n\}$ is a sequence of measurable functions on D that converges to f in measure, show that $\{\phi(x, f_n(x))\}$ converges to $\phi(x, f(x))$ in measure.

Question 4 [10 marks]

Find the function $I(y) = \int_0^\infty [e^{-ax^2} \cos(yx)]dx$ if $a > 0$ is a constant. Justify your answer.

Question 5 [10 marks]

Compute the integral $\int_0^\pi \frac{x \sin x}{1+a^2-2a\cos x} dx$ where $a > 0$ is a constant. (Hint: Apply Residue theorem to the function $f(z) = z/(a - e^{-iz})$ in a suitable region).

Question 6 [10 marks]

Suppose $f(z)$ is a holomorphic function on the complex plane C. If f locally keeps the area invariant, what will the function f be? (Hint: f locally keeps the area invariant if for each point z_0 in C and any neighborhood Ω of z_0, the area of $f(\Omega)$ is equal to the area of Ω.)

Question 7 [10 marks]

Is there an analytic function f on Δ (unit disk in the complex plane with center 0) such that $|f(z)| < 1$ for $|z| < 1$ with $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$? If so, find such an f. Is it unique?

Question 8 [10 marks]

Let $m < n$ be two positive integers and Ω and G be open subsets in R^n and R^m respectively. Does there exist a map $f : \Omega \to G$ such that f and the inverse of f are both C^1? Justify your answer.

Question 9 [10 marks]

Is there a square integrable function f on $[0, \pi]$ such that both inequalities

$$\int_0^\pi (f(x) - \sin x)^2 dx \leq \frac{4}{9}$$

and

$$\int_0^\pi (f(x) - \cos x)^2 dx \leq \frac{1}{9}$$

hold? Justify your answer.

Question 10 [10 marks]

Let α_k for $k = 1, 2, \cdots, n$ be n real numbers such that $0 < \alpha_k < \pi$ for any k. Define $\alpha = \frac{1}{n}\sum_{k=1}^n \alpha_k$. Show that

$$(\prod_{k=1}^n \frac{\sin \alpha_k}{\alpha_k})^{1/n} \leq \frac{\sin \alpha}{\alpha}.$$

END OF PAPER