PAPER 1 — ALGEBRA

Answer all questions. Each question carries 20 marks.

(1) Let G be a group of order pqr, where $p > q > r$ are distinct prime integers.
 (a) Show that G has a normal subgroup of order p.
 (b) Show that if $p \not\equiv 1 \pmod{q}$ and $p, q \not\equiv 1 \pmod{r}$, then G is cyclic.
 (c) Show that if $p \equiv 1 \pmod{q}$ or $p \equiv 1 \pmod{r}$ or $q \equiv 1 \pmod{r}$, then G may be non-Abelian.

(2) Let α and β be linear operators on a finite-dimensional vector space V over an arbitrary field F. Let $\gamma = \beta \circ \alpha$ and $\delta = \alpha \circ \beta$. Prove that if γ is diagonalisable, then γ^2 and δ^2 are conjugates, i.e. there exists a bijective linear operator ψ on V such that $\psi^{-1} \circ \gamma^2 \circ \psi = \delta^2$.

(3) Let R be the subring of $\mathbb{Q}[X]$ consisting of all polynomials with integer constants, i.e.

$$R = \{\sum_{i=0}^{n} a_i X^i \mid n \in \mathbb{Z}_{\geq 0},\ a_0 \in \mathbb{Z},\ a_1, a_2, \ldots, a_n \in \mathbb{Q}\}.$$

 (a) Show that R does not satisfy the ascending chain conditions for principal ideals.
 (b) Determine the units of R.
 (c) Let $p(X)$ be an irreducible element of R.
  (i) Prove that either $p(X) = a_0$ where a_0 a prime integer, or $p(X) = \sum_{i=0}^{n} a_i X^i$ is irreducible in $\mathbb{Q}[X]$ with $a_0 = \pm 1$.
  (ii) Hence, or otherwise, show that $p(X)$ is prime in R.

(4) Let K, L and M be fields, with $K \subseteq L \subseteq M$. Prove or disprove each of the following statements:
 (a) If L is a finite extension over K, and M is a finite extension over L, then M is a finite extension over K.
 (b) If L is an algebraic extension over K, and M is an algebraic extension over L, then M is an algebraic extension over K.
 (c) If L is a separable extension over K, and M is a separable extension over L, then M is a separable extension over K.
 (d) If L is a normal extension over K, and M is a normal extension over L, then M is a normal extension over K.
 (e) If L is a Galois extension over K, and M is a Galois extension over L, then M is a Galois extension over K.

(5) Let R be a ring with multiplicative identity. Consider the following commutative diagram of R-modules and R-module homomorphisms:

$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{\phi_1} & M_2 & \xrightarrow{\phi_2} & M_3 & \xrightarrow{\phi_3} & M_4 & \xrightarrow{\phi_4} & M_5 \\
\alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \alpha_4 \downarrow & & \alpha_5 \downarrow \\
N_1 & \xrightarrow{\psi_1} & N_2 & \xrightarrow{\psi_2} & N_3 & \xrightarrow{\psi_3} & N_4 & \xrightarrow{\psi_4} & N_5
\end{array}$$

(hence, for example, $\psi_3 \circ \alpha_3 = \alpha_4 \circ \phi_3$).

(a) Suppose that α_i's are bijective. Show that if the top row is exact, then so is the bottom row.

(b) Suppose that both rows are exact, and $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are bijective. Show that α_3 is bijective.