NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER II EXAMINATION 2008-2009

MA5206 Graduate Analysis II

April/May 2009— Time allowed : 2 and 1/2 hours

INSTRUCTIONS TO CANDIDATES

(1) This examination paper consists of FIVE questions.

(2) Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

(3) All theorems and results used in your answers should be clearly stated.

Answer **all** the questions.

Question 1 [12 marks]

Let K be a compact convex set in a real locally convex topological vector space and $x_0 \notin K$. Show that there exists a continuous linear functional l and a real constant c such that $l(x_0) > c > l(x)$ for all $x \in K$.

Question 2 [22 marks]

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^3$. Let

$$J[u] = \int_{\mathcal{D}} |\nabla u|^2 dx + \int_{\mathcal{D}} |xu|^3 dx \quad \text{for any } u \in W^{1,2}(\mathcal{D}).$$

Show that there exists $u_0 \in W_0^{1,2}(\mathcal{D})$, $\|u_0\|_{L^4(\mathcal{D})} = 1$ such that

$$J[u_0] = \inf\{J[u] : u \in W_0^{1,2}(\mathcal{D}), \|u\|_{L^4(\mathcal{D})} = 1\}.$$

Let $\{a_{ij}\}_{i,j=1}^3$ be a collection of bounded functions on $\mathcal{D}$ such that

$$|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \quad \text{for any } x \in \mathcal{D}, \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

Does the above conclusion hold for

$$J[u] = \int_{\mathcal{D}} \sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx \, ?$$

Justify your answer.

Question 3 [14 marks]

Let $\mathcal{D}$ be a bounded domain and $1 \leq q < \infty$. Suppose there exists $C_1 > 0$ such that

$$\|f - f_{\mathcal{D}}\|_{L^q(\mathcal{D})} \leq C_1 \|\nabla f\|_{L^2(\mathcal{D})} \qquad (*)$$

for all $f \in C^\infty(\mathcal{D}) \cap W^{1,2}(\mathcal{D})$ where $f_{\mathcal{D}} = \int_{\mathcal{D}} f / |\mathcal{D}|$. Show that $(*)$ holds for all $f \in W^{1,2}(\mathcal{D})$.

Question 4 [40 marks]

Prove or disprove FIVE (5) of the following statements.

(a) If $T : X \to X$ is a compact linear map and X is an infinite dimensional Banach space, then 0 is an eigenvalue.

(b) If $T : X \to X$ is a compact linear map and X is an infinite dimensional Banach space, then for any $y \in X$ and $\lambda \neq 0$ such that λ is not an eigenvalue, there exists a unique $x \in X$ such that $Tx = y + \lambda x$.

(c) Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$, $1 < p < 2$ and let X be a Banach space. If $T : X \to L^2(\mathcal{D})$ is such that $T : X \to L^1(\mathcal{D})$ is compact and $T : X \to L^2(\mathcal{D})$ is bounded linear, then $T : X \to L^p(\mathcal{D})$ is also compact.

(d) The sequence $\{\sin(kx)\}$ has a subsequence that converges weakly in $L^9[0, \pi]$.

(e) Let $\mathcal{P}_3(\mathbb{R}^n)$ be the space of polynomials of degree less than 3 on $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that

$$\|p\|_{L^\infty(B_2(0))} \leq C\|p\|_{L^1(B_1(0))}$$

for all polynomial $p \in \mathcal{P}_3(\mathbb{R}^n)$ where

$$B_r(0) = \{x \in \mathbb{R}^n : |x| < r\} \quad \text{for any } r > 0.$$

(f) If $Tg(x) = \int \sum_{k=1}^{\infty} \frac{t}{k} \sin(kx) g(t) dt$ for any $g \in L^2[0, \pi]$, then $T : L^2[0, \pi] \to L^2[0, \pi]$ is a compact linear operator.

Question 5 [12 marks]

Let $1 < p < \infty$. Let $\{T_k\}$ be a bounded sequence of linear operators from $L^2(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Suppose for each continuous function f with compact support on $\mathbb{R}^n$, there exists $f_0 \in L^p(\mathbb{R}^n)$ such that $\int T_k(f)g \to \int f_0 g$ for all Lipschitz continuous functions g with compact support on $\mathbb{R}^n$. Show that there exists a bounded linear operator T_0 from $L^2(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ such that T_k converges to T_0 weakly.

END OF PAPER