NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2008-2009

# MA5204 Graduate Algebra II

May 2009 – Time allowed : 2.5 hours

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*INSTRUCTIONS TO CANDIDATES*

1. This examination paper contains **FIVE (5)** questions and comprises **SIX (6)** printed pages.
2. Answer not more than **FOUR (4)** questions.
3. Maximum marks will be allocated as follows:

| | | | |
|---|---|---|---|
| Your best 3 answers | : | 28% each | 84% |
| Next best answer | : | 16% | 16% |
| | | | --- |
| | | | 100% |

4. Candidates may use their own notes, but not published material.
5. Results proved in lectures or tutorial assignments that you use should be stated clearly but need not be proved.
6. The symbol $R$ always refers to a ring (with 1), assumed arbitrary, unless otherwise stated.

**Question 1.**

**(a)** Let ORD denote the category of (totally) ordered sets whose morphisms are the order-preserving functions. Let $^*\text{FREE}_R$ denote the category of based free (right) $R$-modules: an object is a pair $(V, \mathcal{F}_\Lambda)$ where $V$ is free and $\mathcal{F}_\Lambda$ is a basis ordered by the ordered set $\Lambda$, and a morphism from $(V', \mathcal{F}'_{\Lambda'})$ to $(V, \mathcal{F}_\Lambda)$ is an $R$-homomorphism from $V'$ to $V$ that restricts to an order-preserving map from $\mathcal{F}'_{\Lambda'}$ to $\mathcal{F}_\Lambda$. Let Fr :ORD$\to$ $^*\text{FREE}_R$ associate with $\Lambda$ the standard free module $R^\Lambda$ with standard basis $\mathcal{E}_\Lambda = (e_\lambda)_{\lambda\in\Lambda}$ and let Ord : $^*\text{FREE}_R \to$ ORD send $(V, \mathcal{F}_\Lambda)$ to the ordered set $\Lambda$.

Show that $^*\text{FREE}_R$ and ORD are equivalent categories, with $\text{Ord}\circ\text{Fr} = \text{Id}$ but $\text{Fr}\circ\text{Ord} \neq \text{Id}$.

**(b)** A commutative ring $R$ is called a *GE-ring* (generalized Euclidean) if for all $n \geq 2$ we have $E_n(R) = \text{SL}_n(R)$, where $E_n(R)$ is the subgroup of $\text{GL}_n(R)$ generated by all elementary $n \times n$ matrices over $R$. Prove that this is equivalent to the condition that for all $n \geq 2$ every matrix in $\text{GL}_n(R)$ can be written as the product of one diagonal matrix and a number of elementary matrices. [You may use without proof the facts that:
(i) for any $A \in \text{GL}_n(R)$, the $2n \times 2n$ matrix

$$\begin{bmatrix} A & O \\ O & A^{-1} \end{bmatrix}$$

lies in $E_{2n}(R)$; and
(ii) for any permutation matrix $P \in \text{GL}_n(R)$ we have $PE_n(R)P^{-1} = E_n(R)$.]

**(c)** Let $R$ be a commutative ring in which 2 is invertible, and let $V$ be a free $R$-module of finite rank. With $\text{Skew}_R(V)$, $\text{Bilin}_R(V)$ and $\text{Quad}_R(V)$ being respectively the $R$-modules of skew-symmetric forms, all bilinear forms and quadratic forms on $V$, show that there is a short exact sequence

$$0 \to \text{Skew}_R(V) \longrightarrow \text{Bilin}_R(V) \longrightarrow \text{Quad}_R(V) \to 0.$$

Does it split?

**Question 2.**

**(a)** The *radical series* of a module $M$ is defined by

$$\mathrm{rad}^0(M) = M \text{ and } \mathrm{rad}^{i+1}(M) = \mathrm{rad}(\mathrm{rad}^i(M)) \text{ for } i \geq 1.$$

Similarly, define the socle series

$$0 \subseteq \mathrm{soc}(M) = \mathrm{soc}^1(M) \subseteq \mathrm{soc}^2(M) \subseteq \cdots \subseteq M,$$

where $\mathrm{soc}^{i+1}(M)$ is defined to be the inverse image of $\mathrm{soc}(M/\mathrm{soc}^i(M))$ in $M$.

Choosing one of these two series, show that it is a fully invariant series for $M$ (meaning that each term is fully invariant).

**(b)** Let $M$ be an Artinian module.

(i) Show that the following are equivalent.

($\alpha$) $\mathrm{soc}(M) = M$; that is, $M$ is semisimple.

($\beta$) $\mathrm{rad}(M) = 0$.

($\gamma$) Every quotient module of $M$ is semisimple.

(ii) Prove that $M/\mathrm{rad}(M)$ is semisimple.

(iii) Show that if $\mathrm{rad}^k(M) = 0$ then $\mathrm{soc}^k(M) = M$.

**(c)** Describe the radical series and socle series of the finitely generated Artinian $\mathbb{Z}$-module

$$M = \mathbb{Z}/p_1^{r_1} \oplus \cdots \oplus \mathbb{Z}/p_s^{r_s}\,.$$

**Question 3.**

**(a)**(i) Show that for any object $C$ of an abelian category $\mathcal{C}$, the functor $\mathrm{Hom}(C,-) : \mathcal{C} \to$ AB is left exact; that is, if

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

is exact in $\mathcal{C}$, then

$$0 \to \mathrm{Hom}(C, M') \xrightarrow{\alpha_*} \mathrm{Hom}(C, M) \xrightarrow{\beta_*} \mathrm{Hom}(C, M'')$$

is exact in AB.

(ii) In the case of a group $G$ with $\mathcal{C} = {}_{\mathbb{Z}[G]}$MOD and $C = \mathbb{Z}$ with trivial $G$-action, show that this $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$ sequence is exact in AB if and only if $\alpha_* : H^1(G; M') \to H^1(G; M)$ is injective.

**(b)** In the other direction, suppose that morphisms $\alpha : M' \to M$ and $\beta : M \to M''$ of an abelian category $\mathcal{C}$ have the property that for every object $C$ , the sequence

$$\mathrm{Hom}(C, M') \xrightarrow{\alpha_*} \mathrm{Hom}(C, M) \xrightarrow{\beta_*} \mathrm{Hom}(C, M'')$$

is exact in AB. By making appropriate choices for $C$, deduce that the sequence

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

is exact in $\mathcal{C}$.

**(c)** Show that if $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are additive functors between abelian categories such that $L$ is left adjoint to $R$, then $R$ is a left exact functor. By considering opposite functors, deduce that $L$ is a right exact functor.

**(d)** Show that for any small category $\mathcal{I}$, if $\mathcal{C}$ is an abelian category for which the colimit functor Colim: $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ exists, then Colim is a right exact functor.

**(e)** Show that the pushout of abelian groups provides an example where Colim:$\text{AB}^{\mathcal{I}} \to$ AB is not an exact functor.

**Question 4.**

We consider a *local* ring $R$ (not necessarily commutative), that is, $R$ has a unique maximal ideal.

**(a)** Show that

(i) the Jacobson radical $\mathrm{rad}(R)$ consists of all the nonunits: $\mathrm{rad}(R) = R - U(R)$, and

(ii) in $R$ the sum of a unit and a nonunit is always a unit.

**(b)** Let us call $M \in M_n(R)$ *unit-diagonal* if each diagonal entry is a unit, and each off-diagonal entry is a nonunit.

(i) Show that every unit-diagonal matrix may be row-reduced by elementary row operations to an upper triangular unit-diagonal matrix.

(ii) From (i) and then its column-reduction counterpart, deduce that every unit-diagonal matrix is invertible.

**(c)** Denote by $M_n(\pi) : M_n(R) \twoheadrightarrow M_n(R/\mathrm{rad}(R))$ the homomorphism induced from the canonical ring projection $\pi : R \twoheadrightarrow R/\mathrm{rad}(R)$.

(i) Show that $M_n(\pi)^{-1}(I_n) \subseteq \mathrm{GL}_n(R)$.

(ii) Show that $M_n(\pi)^{-1}\mathrm{GL}_n(R/\mathrm{rad}(R)) = \mathrm{GL}_n(R)$.

**(d)** Let $P, Q$ be right $R$-modules such that there is an isomorphism $\varphi : P \oplus Q \xrightarrow{\cong} R^n$, and choose elements $p_1, \ldots, p_k \in P$ and $q_1, \ldots, q_{n-k} \in Q$ such that the $\pi(p_i)$ and $\pi(q_j)$ together form a basis for a vector space of dimension $n$ over the division ring $R/\mathrm{rad}(R)$. By considering matrices for linear transformations, deduce that the $p_i$ form a basis for $P$ as a free $R$-module.

**(e)** Calculate $K_0(R)$.

**(f)** Now suppose that also $R$ is a Dedekind domain. What can you deduce from (e) about the number of generators required for each ideal of $R$?

**Question 5.**

**(a)** Let $B$ be a left $R$-module over a ring $R$. Prove that the following are equivalent.

(i) $B$ is flat.

(ii) For every right $R$-module $A$ and every $n \geq 1$, $\mathrm{Tor}_n^R(A, B) = 0$.

(iii) For every right $R$-module $A$, $\mathrm{Tor}_1^R(A, B) = 0$.

**(b)** Suppose that

$$0 \to B' \longrightarrow B \longrightarrow B'' \to 0$$

is a short exact sequence of left $R$-modules.

(i) Suppose that $B''$ is flat. Prove that $B'$ is flat if and only if $B$ is flat.

(ii) Give an example where $B'$ and $B$ are flat, but $B''$ is not flat.

**(c)** Show that if a right $R$-module $A$ has homological dimension (also known as projective dimension) less than $n$, then $\mathrm{Tor}_n^R(A, B) = 0$.

**(d)** Show that over a principal ideal domain every torsion-free module is the direct limit of free modules, and so flat. [You may use without proof the fact that every finitely generated torsion-free module over a pid is free.]

**(e)** Find the minimum value of $k$ such that, for all abelian groups $A, B$, $\mathrm{Tor}_k^{\mathbb{Z}}(A, B) = 0$.

**END OF PAPER**