NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2008-2009

MA3501 Mathematical Methods in Engineering

April/May 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **SEVEN (7)** printed pages, including three appendices: Formulae, Tables of Standard Normal Distribution and Boundary Value Problems.

2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Three appendices P5-P7: Formulae, Tables of Standard Normal Distribution and Boundary Value Problems.

Question 1 [10 marks]

(a) From long experience with a process for manufacturing an alcoholic beverage it is known that the yield is normally distributed with a mean of 500 and a standard deviation of 96 units. For a modified process the yield is 535 units for a sample of size 50. At 5% level of significance, does the modified process increase the yield?

(b) In measuring reaction time, a psychologist knows from a previous survey that the standard deviation is 0.05 seconds. How large a sample of measurements must he take in order to be 95% confident that the error in his estimate of mean reaction time will not exceed 0.01 seconds?

Question 2 [10 marks]

Solve, by the method of characteristics, the following 1^{st} order PDE:

$$\begin{array}{rl} u_t(x,t) + 2\sin t \; u_x(x,t) = x & , \; -\infty < x < \infty, \; t > 0 \\ u(x,0) = x^2 & , \; -\infty < x < \infty \end{array}$$

Question 3 [16 marks]

(a) Solve the following heat equation by the method of separation of variables:

$$\begin{array}{rl} u_t(x,t) = c^2 u_{xx}(x,t) & , \; -\pi < x < \pi, \; t > 0, \\ u(-\pi,t) = u(\pi,t), \; u_x(-\pi,t) = u_x(\pi,t) & , \; t > 0, \\ u(x,0) = f(x) & , \; -\pi < x < \pi. \end{array}$$

(You don't need to compute the coefficients A_n, B_n given in the table of boundary value problems)

(b) A thin circular ring of unit radius is insulated along its lateral sides. Suppose the initial temperature of the ring is given by $f(x) = \frac{100}{\pi}(x+\pi)$, where x represents arclength along the ring, $-\pi < x \le \pi$. Find the steady-state (final) temperature of the ring.

Question 4 [16 marks]

(a) Use the method of the Fourier transform with respect to x to solve the following PDE:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad , \quad -\infty < x < \infty,\ y > 0, \\ u(x,0) &= f(x) \quad , \quad -\infty < x < \infty. \end{aligned}$$

We assume that the Fourier transform $\mathfrak{F}(u(x,t)) = \hat{u}(\omega, t)$ is bounded.

(Present $\hat{u}(\omega, t)$ in terms of exponential functions. The solution is of integral form. You don't need to simplify your solution.)

(b) In the above PDE, suppose that $f(x) = 100$ if $-1 < x < 1$ and $f(x) = 0$ otherwise. Find the equation of the curve such that $u(x, y) = 50$.

(You **need** to simplify the equation of the curve.)

Question 5 [16 marks]

Suppose that a chemical substance is being poured at a constant rate α into a straight narrow, clean river that flows with a constant velocity v_0. Assume that the river is very fast. Then the concentration $u(x, t)$ of the substance at a distance x downstream at time t is governed by

$$\begin{aligned} u_t &= -v_0 u_x \quad , \quad x > 0,\ t > 0, \\ u(0,t) &= \alpha \quad , \quad t > 0, \\ u(x,0) &= 0 \quad , \quad x > 0. \end{aligned}$$

Use the method of Laplace transform with respect to t to solve the above PDE.

At the point X, find the time T such that the concentration $u(X, t)$ of the substance remains constant when $t \geq T$. What is the steady-state (final) concentration of the river?

Question 6 [12 marks]

(a) Evaluate $\displaystyle\int_\gamma \left(\frac{z+2}{z} + \frac{z^3+3}{(z-i)^2} + \frac{\sin z}{z-3}\right) dz$

where γ is a circle with centre $(0, 0)$ and radius 2 in anticlockwise direction.

(b) It is known that $f(z) = Logz = ln \mid z \mid + iArgz$ is analytic in the upper-half plane (without x-axis).Use a property of this analytic function $f(z)$ to prove that

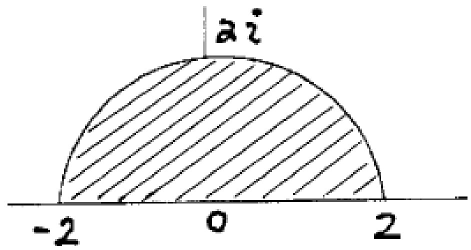
$$u(x,y) = \frac{50}{\pi}\tan^{-1}\frac{y}{x-3} - \frac{50}{\pi}\tan^{-1}\frac{y}{x-1} + 50$$

is a solution of the following PDE:

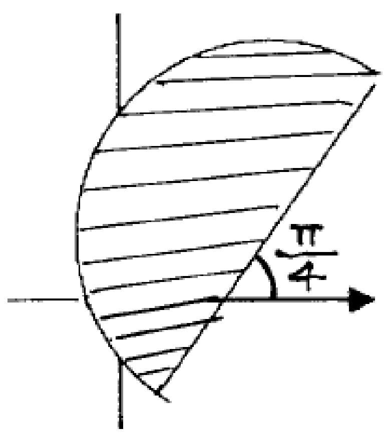
$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad , \quad -\infty < x < \infty,\ y > 0, \\ u(x,0) &= 50 \quad , \quad x > 3, \\ u(x,0) &= 0 \quad , \quad 1 < x < 3, \\ u(x,0) &= 50 \quad , \quad x < 1. \end{aligned}$$

Question 7 [20 marks]

(a) Find the image of the following upper half of the disk with centre $(0,0)$ and radius 2 under the mapping $w = \dfrac{z-2}{z+2}$.



(b) Find a mapping $f(z)$ which maps the following semicircle with centre $1+i$ and radius 2



to the lower half-plane $\{(x, y) : y < 0\}$.

(you don't need to simplify you function f(z).)

END OF PAPER