MATRICULATION NO.: ______________________________

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2008-2009

MA2214 Combinatorial Analysis

May 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **TWELVE (12)** questions and comprises **TEN (10)** printed pages.
2. Answer **ALL** the questions. The points for the questions are not necessarily the same; points for each question are indicated at the beginning of the question.
3. Please write your combinatorial expressions using only factorials or terms of the form $\binom{n}{k}$, but do not simplify those any further.
4. A total of 100 points are possible.

Question 1 [9 points] In how many ways can three tickets be distributed among twenty students if

(i) the tickets are for different shows, and each student can get no more than one ticket? (3 points)

(ii) the tickets are for different shows, and each student can get any number of tickets (up to three)? (3 points)

(iii) the tickets are for the same show, and each student can get no more than one ticket? (3 points)

Question 2 [6 points] Prove that $R(3, 4) > 7$.

Question 3 [15 points] In how many ways can you permute the letters of the English alphabet so

(i) none of the sequences *dog*, *rat*, and *finch* appear? (5 points)

(ii) at least one of these sequences appears? (5 points)

(iii) exactly one of these sequences appears? (5 points)

Question 4 [5 points] Suppose a farm has 9 alpacas, 20 sheep, and 4 llamas. What is the smallest number of animals that you must choose to make sure that you have at least 3 of some type of animal? Explain your answer.

Question 5 [5 points] A cryptologist is given a list of 20,000 codes, each of which is a sequence of length 4 or less containing only the numbers $0, 1, 2, \ldots, 9$ (repetitions and leading 0s are allowed). Is it possible for all of the 20,000 codes to be distinct? Explain your answer.

Question 6 [10 points] Use Ferrers diagrams to prove that the number of partitions of n into parts of even size is equal to the number of partitions of n into parts such that parts of a given size occur an even number of times.

Question 7 [10 points] Prove that the generating function for $a_n = \binom{2n}{n}$ is

$$A(x) = \frac{1}{\sqrt{1-4x}}.$$

Question 8 [3 points] Consider the generating function $A(x) = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}$. The coefficient of x^6 in the expansion of $A(x)$ is 7. What does this tell you about partitions of the number 6?

Question 9 [10 points] A ship carries 36 flags: 12 red, 12 white, and 12 black. Twelve of these flags are placed on a vertical pole to signal to other ships, and any such order is a signal.

(i) How many of these signals use an even number of red flags and an even number of black flags? (5 points)

(ii) How many of these signals use at least three white flags or no white flags at all? (5 points)

Question 10 [10 points] Solve the following recurrence relations.

(i) $a_{n+2} - 4a_{n+1} - 5a_n = 8n$, where $a_0 = 1$ and $a_1 = 9$. (5 points)

(ii) $a_{n+2} + a_{n+1} - 6a_n = 10 \cdot 2^n$, where $a_0 = 2$ and $a_1 = 1$. (5 points)

Question 11 [5 points] Solve the recurrence relation $a_{n+1} - a_n = 3$ when $a_0 = 5$ by the method of generating functions.

Question 12 [12 points]

(i) How many ways are there to place k objects in a line if each of the objects is of one of n types and each type of object can appear any number of times? (2 points)

(ii) Prove your answer using exponential generating functions. (10 points)

END OF PAPER