NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2008-2009

MA2213 Numerical Analysis I

May 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination.

2. This examination paper contains a total of **SIX (6)** questions and **ONE (1)** appendix and comprises **FIVE (5)** printed pages.

3. Answer **ALL** questions in **Section A**. Each question in Section A carries 20 marks.

4. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1. [20 marks]

(a) Solve the following linear system of equations by Gaussian elimination with partial pivoting and four-digit rounding arithmetic.

$$\begin{aligned} 2.000x + 0.6525y &= 5.200, \\ 3.000x - 4.000y &= 3.000. \end{aligned}$$

(b) Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 2 \\ 2 & 0 & 5 \end{bmatrix}.$$

Factor matrix A into the form $A = LU$, where L is a lower triangular matrix whose diagonal entries are all one, and U is an upper triangular matrix.

Question 2. [20 marks]

(i) Use initial guess $p_0 = 3$ and do one step of Newton's method to obtain p_1 to estimate the value of $\sqrt[3]{40}$.

(ii) Suppose one more step of Newton's iteration is performed and the absolute errors in approximations p_1, p_2 with the exact value p are given as follows.

$$|p_1 - p| = 6.15 \times 10^{-2},$$
$$|p_2 - p| = 1.08 \times 10^{-3}.$$

Without calculating the next approximation p_3, which one of the three values listed below is most likely to be the value of $|p_3 - p|$? Explain why.

$$r_1 = 1.89050847 \times 10^{-5},$$
$$r_2 = 3.41564518 \times 10^{-7},$$
$$r_3 = 3.41948692 \times 10^{-14}.$$

. . .–3–

Question 3. [20 marks]

(i) Denote by T_n the Composite Trapezoidal rule approximating

$$\int_0^2 \frac{1}{x+2} dx$$

with n subintervals. Find T_1, T_2 and T_4.

(ii) Use the three values T_1, T_2 and T_4 obtained in (i) to approximate the integral $\int_0^2 \frac{1}{x+2} dx$ as accurately as possible.

SECTION B

Answer not more than **two** *questions from this section. Section B carries a total of 40 marks.*

Question 4. [20 marks]

Suppose the polynomial that interpolates function $f(x)$ at the points $x_0 = 0$,$x_1 = 1$, $x_2 = 2$ is given by

$$P(x) = 0.2426x^2 - 0.8344x + 1.0000.$$

(i) Find the polynomial that interpolates the function $F(x) = f(x) + 0.0023x^2 + xe^{-x}$ at the points x_0, x_1 and x_2.

(ii) It turns out that a function of form

$$y = \frac{1}{\sqrt{ax+b}}$$

can approximate the function $F(x)$ better. Using the three points given below, do a least squares approximation to determine the values of a and b by first transforming the problem to a linear least squares problem.

x	0	0.5	1.5
$F(x)$	1	0.84	0.68

. . .–4–

Question 5. [20 marks]

(a) If we approximate the integral $\int_a^b f(x)dx$ by the Composite Trapezoidal rule with n subintervals, $T(n)$, we obtain

$$T(24) = 0.80326, \quad T(48) = 0.80440, \quad T(96) = 0.80468.$$

Use this information to compute the Composite Simpson's rule estimates with n subintervals, $S(n)$, for $n = 48$ and $n = 96$.

(b) Determine a, b and c such that the quadrature formula

$$Q(f) = af(0) + bf(\frac{1}{2}) + cf(1)$$

is exact for the integral

$$\int_0^1 f(x)x^2dx,$$

if $f(x)$ is a polynomial of degree 2 or less, that is, is exact for $f(x) = 1, f(x) = x$ and $f(x) = x^2$. Then use this quadrature formula to approximate the integral

$$\int_{-1}^1 x^2(x+1)^2\sin(x+1)dx.$$

Question 6. [20 marks]

Let $P_n(x)$ be the polynomial of degree n that interpolates a function $f(x)$ at $n+1$ points $x_0, x_1, \cdots, x_n$.

(i) Show that

$$f(x) = P_n(x) + f[x_0, x_1, \cdots, x_n, x](x - x_0)(x - x_1)\cdots(x - x_n).$$

(ii) Given

$$x_0 = -1, \qquad x_1 = 1, \qquad x_2 = 2, \qquad x_3 = 4,$$

and the interpolation polynomial $P_3(x)$, find an error bound for the absolute error in $P_3(0)$ as an approximation to $f(0)$. Suppose $|f[-1, 1, 2, 4, x]| \leq 0.01$ for all x in the interval $[-1, 4]$.

END OF PAPER

Appendix: MA2213 Formula Sheet (One page)

- Significant digits is the largest nonnegative integer k satisfying
$$\frac{|p-p^*|}{|p|} \le 0.5 \times 10^{1-k}.$$
- Root-finding methods:
  - Bisection: $p_n = \frac{a_n+b_n}{2}$, $|p_n - p| \le \frac{b-a}{2^n}$;
  - Newton's: $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$;
  - Modified Newton's: set $\mu(x) = \frac{f(x)}{f'(x)}$;
  - Secant: $p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1}-p_{n-2})}{f(p_{n-1})-f(p_{n-2})}$.
- Order of convergence α: $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$.
- Interpolation polynomial:
  - Lagrange:
$$P_n(x) = \sum_{k=0}^{n} f(x_k)L_k(x),\ L_k(x) = \prod_{i=0, i\neq k}^{n} \frac{(x-x_i)}{(x_k-x_i)}.$$
  - Newton's divided difference:
$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0,\cdots,x_k](x-x_0)\cdots(x-x_{k-1}).$$
$$f[x_i,\cdots,x_{i+k}] = \frac{f[x_{i+1},\cdots,x_{i+k}] - f[x_i,\cdots,x_{i+k-1}]}{x_{i+k}-x_i}.$$
  - Error estimate:
$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i).$$
- Hermite interpolation:
$$\begin{aligned} H(x) &= \sum_{j=0}^{n} f(x_j)H_j(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_j(x),\\ H_j(x) &= [1-2(x-x_j)L_j'(x_j)]L_j^2(x),\\ \hat{H}_j(x) &= (x-x_j)L_j^2(x). \end{aligned}$$
$$f(x) = H(x) + \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\prod_{i=0}^{n}(x-x_i)^2.$$
- Piecewise linear interpolation:
$$I_1(x) = \sum_{k=0}^{n} f(x_k)l_k(x),\ l_k(x) = \begin{cases} \frac{x-x_{k-1}}{x_k-x_{k-1}}, & x_{k-1}\le x\le x_k,\\ \frac{x-x_{k+1}}{x_k-x_{k+1}}, & x_k \le x \le x_{k+1},\\ 0, & \text{otherwise.}\end{cases}$$
- Cubic spline interpolation:
$$\begin{aligned} &S_j(x_{j+1}) = f(x_{j+1}) = S_{j+1}(x_{j+1});\\ &S_j'(x_{j+1}) = S_{j+1}'(x_{j+1});\\ &S_j''(x_{j+1}) = S_{j+1}''(x_{j+1});\\ &S''(x_0) = S''(x_n) = 0 \text{ (free)};\\ &S'(x_0) = f'(x_0),\ S'(x_n) = f'(x_n) \text{ (clamped)}. \end{aligned}$$
- Numerical integration:

Rule		Error
Trapezoidal	–	$-\frac{h^3}{12}f''(\xi)$
Simpson's	$\frac{h}{3}(f_0+4f_1+f_2)$	$-\frac{h^5}{90}f^{(4)}(\xi)$
Simpson's 3/8	$\frac{3h}{8}(f_0+3f_1+3f_2+f_3)$	$-\frac{3h^5}{80}f^{(4)}(\xi)$
Boole's	$\frac{2h}{45}(7f_0+32f_1+12f_2+32f_3+7f_4)$	$-\frac{8h^7}{945}f^{(6)}(\xi)$
Midpoint	–	$\frac{h^3}{24}f''(\xi)$
General	$\sum_{i=0}^{n} a_i f(x_i),\quad a_i = \int_a^b L_i(x)dx$	–
Composite Trapezoidal		$-\frac{b-a}{12}h^2 f''(\xi)$
Composite Simpson's		$-\frac{b-a}{180}h^4 f^{(4)}(\xi)$

- Gaussian quadrature:
$$\int_{-1}^{1} f(x)dx \approx f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}).$$
- Romberg integration:
$$R_{k,j} = \left[R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1}-1}\right],$$
where the error in $R_{k,j}$ is $O(h_k^{2j})$.
- *LU* factorization:
  - Doolittle's method: $l_{ii} = 1$;
$$L = \begin{bmatrix} 1 & 0 & \cdots & 0\\ m_{21} & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix}.$$
  - Crout's method: $u_{ii} = 1$.