NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2008-2009

MA2108 Mathematical Analysis I

April 2009 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring one piece of A4-sized two-sided help sheet into the examination room.

2. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **EIGHT (8)** questions and comprises **FIVE (5)** printed pages.

3. Answer **ALL** questions in **Section A**. Section A carries a total of 70 marks.

4. Answer not more than **TWO (2)** questions from **Section B**. Section B carries a total of 30 marks.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 70 marks.*

Question 1.

Let

$$a_1 = 1, \qquad a_{n+1} = 2 + \frac{7}{a_n + 4}, \qquad \forall n \in \mathbb{N}.$$

Prove that (a_n) converges and find its limit. [10 marks]

Question 2.

(a) Determine whether the following series converge or diverge. Justify your answers.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+5}{1-n^2+2n^3}$. [4 marks]

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{\left(1+\frac{1}{2n}\right)^{4n^2}}$. [4 marks]

(b) Let (a_n) be a sequence of real numbers such that $a_n > 0$ for all $n \in \mathbb{N}$.

(i) Prove that if there exists a natural number K_1 such that

$$\frac{a_{n+1}}{a_n} \geq 1 \qquad \forall n \geq K_1,$$

then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges. [3 marks]

(ii) Prove that if there is a real number r with $0 < r < 1$ and a natural number K_2 such that

$$\frac{a_{n+1}}{a_n} < r \qquad \forall n \geq K_2,$$

then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. [4 marks]

Question 3.

(a) Let (a_n) be a bounded sequence. Prove that (a_n) has a subsequence (a_{n_k}) such that
$$\lim_{k\to\infty} a_{n_k} = \limsup a_n.$$
[6 marks]

(b) Let
$$x_n = \frac{(2n^2+3)\sin(n\pi/4)}{\sqrt{4n^4+5n^3-1}}, \qquad n \in \mathbb{N}.$$

(i) Find $\limsup x_n$ and $\liminf x_n$. [7 marks]

(ii) Is (x_n) convergent? Justify your answer. [2 marks]

Question 4.

(a) Use the $\varepsilon - \delta$ definition of limit to prove that
$$\lim_{x\to 1} \frac{x^2-3x+1}{2x-1} = -1.$$
[7 marks]

(b) In each part, either evaluate the limit or show that the limit does not exist.

(i) $\displaystyle\lim_{x\to 1} x\cos\left(\frac{x}{x-1}\right)$. [4 marks]

(ii) $\displaystyle\lim_{x\to 1} (x-1)^2\cos\left(\frac{x}{x-1}\right)$. [4 marks]

Question 5.

(a) Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x - 1 & \text{if } x \text{ is rational} \\ 2x - 3 & \text{if } x \text{ is irrational.} \end{cases}$$

Find all the point(s) at which f is continuous. Justify your answer. [8 marks]

(b) Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is continuous at the point $x = a$ and $g(a) < M$ for some $M \in \mathbb{R}$. Prove that there exists $\delta > 0$ such that

$$g(x) < M \qquad \forall x \in (a - \delta, a + \delta).$$

[7 marks]

SECTION B

Answer not more than **two** *questions from this section. Section B carries a total of 30 marks.*

Question 6.

(a) Prove that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

[8 marks]

(b) Let the sequence (x_n) be defined by

$$x_1 = 1, \qquad x_{n+1} = x_n + \frac{1}{nx_n}, \qquad n \in \mathbb{N}.$$

Determine whether (x_n) converges. Justify your answer. [7 marks]

Question 7.

(a) Let the function $g : [0,1] \to \mathbb{R}$ be continuous on $[0,1]$. Define $h : [0,1] \to \mathbb{R}$ by

$$h(x) = \sup\{g(t) : \ 0 \le t \le x\}.$$

Prove that h is continuous on $(0,1)$. [7 marks]

(b) Prove that if the function f is continuous on $\mathbb{R}$ and has the property that

$$f(x) = f\left(\frac{x+1}{2}\right)$$

for all $x \in \mathbb{R}$, then f is a constant function on $\mathbb{R}$. [8 marks]

Question 8.

(a) Suppose that the function $f : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and $f(0) = f(1)$. Prove that for each natural number n, there exists a real number x_n such that $0 \le x_n \le 1 - 1/n$ and

$$f(x_n) = f\left(x_n + \frac{1}{n}\right).$$

[8 marks]

(b) Let the function $g : [0,\infty) \to \mathbb{R}$ be continuous on $[0,\infty)$ and

$$\lim_{x\to\infty} g(x) = 1.$$

Prove that g is uniformly continuous on $[0,\infty)$. [7 marks]

END OF PAPER