NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2008-2009

MA2101 Linear Algebra II

April 2009 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections. It contains **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.

3. Answer not more than **TWO (2)** questions in **Section B**. Each question in Section B carries 20 marks.

4. Calculators may be used. However, you should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [15 Marks]

Let $V = \mathcal{P}_3(\mathbb{C})$ and let W_1 and W_2 be subspaces of V such that

$$W_1 = \operatorname{span}\{1 + x^2,\ 1 + 2x^2,\ 1 + 3x^2\}$$

and

$$W_2 = \operatorname{span}\{1 + ix^3\}.$$

Find the dimensions of W_1, W_2, $W_1 \cap W_2$, $W_1 + W_2$ and V/W_1.

Question 2 [15 Marks]

Let T be a linear operator on $\mathcal{M}_{22}(\mathbb{R})$ such that

$$T(\boldsymbol{X}) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \boldsymbol{X} \quad \text{for } \boldsymbol{X} \in \mathcal{M}_{22}(\mathbb{R}).$$

(a) Let $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.

Compute $[T]_B$.

(b) Find the rank and nullity of T.

(c) Is T an isomorphism? Justify your answer.

Question 3 [15 Marks]

Let $\mathbb{C}^4$ be equipped with the usual inner product and let

$$W = \{(a,\ a - bi,\ a + 2bi,\ a + 3bi) \mid a, b \in \mathbb{C}\}.$$

(a) Show that W is a subspace of $\mathbb{C}^4$.

(b) Find an orthonormal basis for W.

(c) Find a formula for the orthogonal projection $\mathrm{Proj}_W : \mathbb{C}^4 \to \mathbb{C}^4$.

(You can express the answer as a linear combination of two vectors.)

Question 4 [15 Marks]

Let Q be a quadratic form on $\mathbb{R}^3$ such that

$$Q((x_1, x_2, x_3)) = (x_1 + x_2 + x_3)^2 \quad \text{for } (x_1, x_2, x_3) \in \mathbb{R}^3.$$

(a) Write down a 3×3 symmetric matrix $\boldsymbol{A}$ so that

$$Q(\boldsymbol{u}) = \boldsymbol{u}\boldsymbol{A}\boldsymbol{u}^{\mathrm{T}} \quad \text{for all } \boldsymbol{u} \in \mathbb{R}^3.$$

(In here, $\boldsymbol{u}$ is written as a row vector.)

(b) Find a 3×3 orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

(c) Find a basis $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ for $\mathbb{R}^3$ so that for all $\boldsymbol{u} = a_1\boldsymbol{v_1} + a_2\boldsymbol{v_2} + a_3\boldsymbol{v_3} \in \mathbb{R}^3$, $a_1, a_2, a_3 \in \mathbb{R}$,

$$Q(\boldsymbol{u}) = \lambda_1 a_1^2 + \lambda_2 a_2^2 + \lambda_3 a_3^2$$

where $\lambda_1, \lambda_2, \lambda_3$ are fixed real numbers.

SECTION B

Answer not more than **two** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 Marks]

Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. Let $P : V \to V$ be a mapping such that $P(\boldsymbol{u}) \in W_1$ and $\boldsymbol{u} - P(\boldsymbol{u}) \in W_2$ for all $\boldsymbol{u} \in V$.
(The mapping P is uniquely defined because $V = W_1 \oplus W_2$.)

(a) Prove that P is a linear transformation.

(b) For $\boldsymbol{v} \in W_1$ and $\boldsymbol{w} \in W_2$, show that $P(\boldsymbol{v}) = \boldsymbol{v}$ and $P(\boldsymbol{w}) = \boldsymbol{0}$.

(c) Let T be a linear operator on V.

(i) Prove that if both W_1 and W_2 are T-invariant, then $T \circ P = P \circ T$.

(ii) If only W_1 is T-invariant, is it true that $T \circ P = P \circ T$?

Question 6 [20 Marks]

(a) Let $V = \mathcal{P}(\mathbb{R})$ be equipped with the following inner product:
For $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, $q(x) = b_0 + b_1 x + \cdots + b_m x^m \in V$,

$$\langle p(x),\, q(x) \rangle = \sum_{i=0}^{\min\{n,m\}} a_i b_i.$$

Let S and T be linear operators on V such that for $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in V$,

$$S(p(x)) = a_0 + (a_1 - a_0)x + \cdots + (a_n - a_{n-1})x^n - a_n x^{n+1},$$
$$T(p(x)) = (a_0 - a_1) + (a_1 - a_2)x + \cdots + (a_{n-1} - a_n)x^{n-1} + a_n x^n.$$

(i) Is S surjective? Is T injective?

(ii) Prove that S is the adjoint of T.

(b) Let T be a linear operator on an inner product space. Let T^* be the adjoint of T.

(i) If T is surjective, prove that T^* is injective.

(ii) If T is injective, must T^* be surjective? Justify your answer.

Question 7 [20 Marks]

(In this question, all vectors are written as column vectors.)

Let $\mathbb{F}$ be a field.

(a) Let $\boldsymbol{J}_t(\lambda) = \begin{pmatrix} \lambda & 1 & & \mathbf{0} \\ & \lambda & 1 & \\ & & \ddots & \ddots & \\ & \mathbf{0} & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \in \mathcal{M}_{tt}(\mathbb{F})$ where $\lambda \in \mathbb{F}$.

(i) Let $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \dots, \boldsymbol{e_t}\}$ be the standard basis for $\mathbb{F}^t$.
For $s, k = 1, 2, \dots t$, compute $(\boldsymbol{J}_t(\lambda) - \lambda \boldsymbol{I_t})^s \boldsymbol{e_k}$.

(ii) For $s \geq 1$, find the nullity of $(\boldsymbol{J}_t(\lambda) - \lambda \boldsymbol{I_t})^s$.

(b) Let $\boldsymbol{A} \in \mathcal{M}_{nn}(\mathbb{F})$ such that $\boldsymbol{A}$ has a Jordan form

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_{t_1}(\lambda) & & & \mathbf{0} \\ & \boldsymbol{J}_{t_2}(\lambda) & & \\ & & \ddots & \\ & & & \ddots & \\ & \mathbf{0} & & & \boldsymbol{J}_{t_m}(\lambda) \end{pmatrix}$$

where $\lambda \in \mathbb{F}$ and $n = t_1 + t_2 + \cdots + t_m$.

(i) Show that for $s \geq 1$, the nullity of $(\boldsymbol{J} - \lambda \boldsymbol{I_n})^s$ is equal to $\displaystyle\sum_{i=1}^{m} \min\{s,\, t_i\}$.

(ii) Suppose $\text{nullity}((\boldsymbol{A} - \lambda \boldsymbol{I_n})^s) - \text{nullity}((\boldsymbol{A} - \lambda \boldsymbol{I_n})^{s-1}) = r$ where $s \geq 2$. What can you say about the sizes of the Jordan blocks $\boldsymbol{J}_{t_i}(\lambda)$ based on the value of r? (You do not need to explain your answer.)

[END OF PAPER]