NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2008-2009

MA1104 Multivariable Calculus

April 2009 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination. Each student is allowed to bring two pieces of A4-sized two-sided help sheets into the examination room.

2. This examination paper consists of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

3. Answer **ALL** questions. Marks for each question are indicated at the end of the question.

4. Candidates may use non-programmable, non-graphic calculators. However, they should lay out systematically the various steps in the calculations.

Answer **ALL** *questions.*

Question 1

(a) Find a parametrization of the curve of the intersection of the surfaces $z = x^2 - y^2$ and $z = x^2 + xy - 1$.

[7 *marks*]

(b) Let c be a scalar, and let $\mathbf{a}$ and $\mathbf{b}$ be three-dimensional vectors. Let $\mathbf{x} = \langle x, y, z \rangle$. Show that the equation $(\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{b}) = c^2$ defines a sphere with center whose position vector is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ and radius R where $R^2 = c^2 + \left\| \frac{1}{2}(\mathbf{a} - \mathbf{b}) \right\|^2$.

[8 *marks*]

Question 2

(a) Determine all points at which the given function is continuous. Justify your answer.

$$g(x, y) = \begin{cases} \frac{x^2 - y^2}{x - y} & \text{if } x \neq y \\ 3x & \text{if } x = y \end{cases}$$

[7 *marks*]

(b) Let

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^6 + 2y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Let $\mathbf{u}$ be any unit vector.

(i) Determine whether the directional derivative $D_{\mathbf{u}} f(0, 0)$ exists. Justify your answer.

(ii) Determine whether f is continuous at $(0, 0)$. Justify your answer.

[8 *marks*]

Question 3

(a) Let C be the curve of intersection of the two surfaces $x^3+2xy+yz = 7$ and $3x^2-yz = 1$. Find the parametric equations of the tangent line to C at $(1, 2, 1)$.

[5 *marks*]

(b) Let $f(x, y) = y^3 - 24x - 3x^2y$. Find all the local extreme points of $f(x, y)$ and classify them.

[5 *marks*]

(c) Find the dimensions of the rectangular box of maximum volume with its sides parallel to the coordinate planes that can be inscribed in the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

[5 *marks*]

Question 4

(a) Suppose $\mathbf{r} = \mathbf{r}(t)$ is a parametrization of the curve which is the intersection of the surfaces $z = \sqrt{4\pi^2 - x^2 - y^2}$ and $x = \sin y$. Let $\mathbf{J} = \mathbf{r} \times \mathbf{r}'$. Determine whether $\mathbf{r}' = (\mathbf{J} \times \mathbf{r})/\left\|\mathbf{r}\right\|^2$. Justify your answers.

[8 *marks*]

(b) Evaluate $\displaystyle\int_0^{1/2} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx.$

[5 *marks*]

(c) Find the volume of the solid Q bounded by $x + y + z = 1$, $x + y + z = 2$, $x + 2y = 0$, $x + 2y = 1$, $y + z = 2$ and $y + z = 4$.

[5 *marks*]

Question 5

(a) Determine whether $\mathbf{F}(x, y, z) = (2x + y)\mathbf{i} + (x + 2yz^2)\mathbf{j} + 2y^2z\mathbf{k}$ is conservative. If it is conservative, find any f such that $\nabla f = \mathbf{F}$. Justify your answers.

[5 *marks*]

(b) Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle x^3 - y^4, e^{x^2+z^2}, x^2 - 16y^2z^2 \rangle$ and C is $x^2 + z^2 = 1$ (counterclockwise when viewed from positive y-axis) in the plane $y = 0$.

[5 *marks*]

(c) Let C be a simple closed curve in the plane $ax + by + cz + d = 0$. Let R be the region enclosed by C and oriented by $\mathbf{n} = \langle a, b, c \rangle$. Prove that the area of R is

$$\frac{1}{2\|\mathbf{n}\|} \oint_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$$

where C is the positively oriented boundary of R.

[7 *marks*]

Question 6

(a) Prove the identity

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \operatorname{curl}(\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot \operatorname{curl}(\mathbf{G})$$

[5 *marks*]

(b) Evaluate $\iint_S y^2\,dS$ where S is that part of the cylinder $x^2 + z^2 = 1$ between the planes $y = 0$ and $y = 3 - x$.

[6 *marks*]

(c) Consider the vector field

$$\mathbf{E} = \nabla \left(\frac{z}{\rho^3} \right)$$

where $\rho = (x^2 + y^2 + z^2)^{1/2}$.

Compute the flux of $\mathbf{E}$ across any smooth closed surface with positive (outward) orientation enclosing the origin.

[9 *marks*]

END OF PAPER