NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION (2008–2009)

MA1102R Calculus

April/May 2009 — Time allowed : 2 hours

INSTRUCTION TO CANDIDATES

1. This examination paper consists of **ONE (1)** section. It contains a total of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [18 marks]

Evaluate the following limits.

(a) $\lim_{x\to\infty}\left(\frac{x^2+2x+3}{4x+5}\cdot\sin\frac{6}{7x}\right)$

(b) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

(c) $\lim_{n\to\infty}\left(\frac{\sqrt[n]{2}+\sqrt[n]{3}+\sqrt[n]{4}}{3}\right)^n$

Question 2 [18 marks]

Evaluate the following definite integrals.

(a) $\int_1^e x^3\ln x\,dx$

(b) $\int_0^1 \tan^{-1}(\sqrt{x})\,dx$

(c) $\int_1^4 \frac{x^2+4x+4}{x^2(x^2+4)}\,dx$

Question 3 [18 marks]

Determine whether each of the following series is convergent or divergent. Justify your answer.

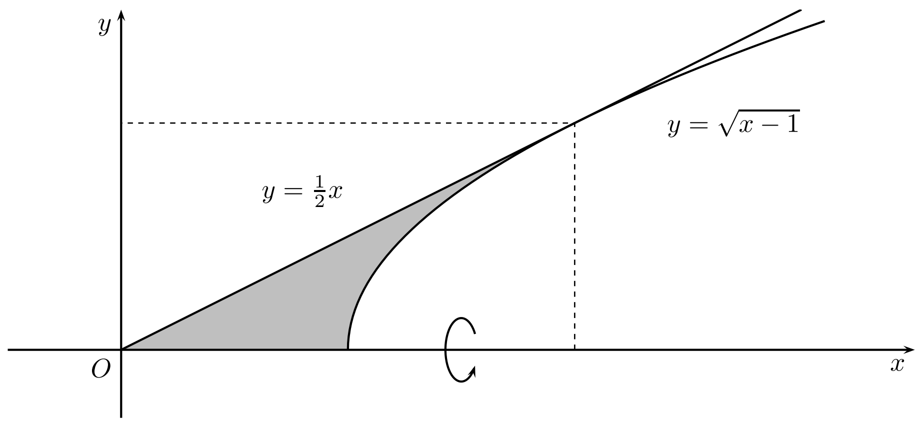
(a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[n]{n^2+1}}$

(b) $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

(c) $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{4\cdot 6\cdot 8\cdots(2n+2)}$

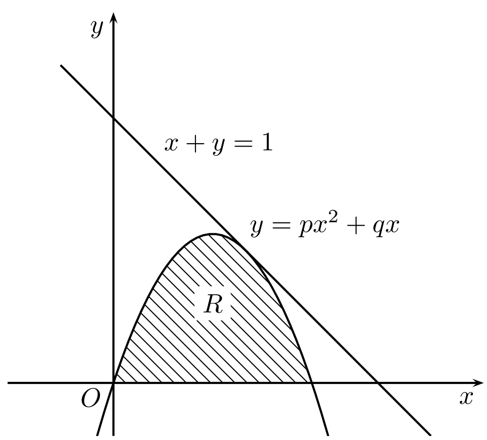
Question 4 [10 marks]

Find the total surface area of the solid obtained by rotating the region bounded by $y = \sqrt{x-1}$, $y = \dfrac{1}{2}x$ and $y = 0$ about the x-axis.



Question 5 [12 marks]

Suppose that the straight line $x + y = 1$ is tangent to the parabola $y = px^2 + qx$, where $p < 0$ and $q > 0$. Let R denote the region bounded by the parabola and the x-axis.



(i) Prove that $4p + (q+1)^2 = 0$.

(ii) Express the area of R in terms of q, and determine the values of p and q when R has the largest area.

Question 6 [12 marks]

(a) Suppose f is a differentiable one-to-one function such that f' is continuous on $\mathbf{R}$. Let $g = f^{-1}$ be the inverse function of f. Let $a, b \in \mathbf{R}$, $c = f(a)$ and $d = f(b)$.

Evaluate the following definite integral

$$\int_a^b f(x)\,dx + \int_c^d g(x)\,dx$$

in terms of a, b, c and d.

(b) Evaluate the series

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}.$$

Question 7 [12 marks]

(a) Let f be a continuous function on $\mathbf{R}$. Suppose that f is differentiable on $\mathbf{R}\backslash\{a\}$ for some real number a, such that $\lim_{x\to a} f'(x)$ exists and equals L.

Prove that f is differentiable at a, and that $f'(a) = L$.

(b) Let $f(x)$ be a non-constant continuous function on $[a, b]$, where $a < b$.

Prove that the range of $f(x)$ on $[a, b]$ is a closed interval $[c, d]$.