Student Number: | U/T/NT* | – | | | – | | | | | |

*Delete where necessary

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2008-2009

MA1101R LINEAR ALGEBRA I

April/May 2009 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper contains a total of **Five (5)** questions and comprises **Twenty Three (23)** printed pages.

3. Answer **ALL** questions. Write your answers and working in the spaces provided inside the booklet following each question.

4. Total marks for this exam is **100**. The marks for each question are indicated at the beginning of the question.

5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Examiner's Use Only	
Questions	Marks
1	
2	
3	
4	
5	
Total	

Question 1 [20 marks]

(a) Let $V = \{(w, x, y, z) \mid y = w - x, z = 2w + x\}$ be a subset of $\mathbb{R}^4$.

(i) [2 marks] Write down an explicit form of a general vector in V.

(ii) [2 marks] Show that V is a subspace of $\mathbb{R}^4$ by expressing V as a linear span.

(iii) [2 marks] Write down a basis for V and $\dim V$.

(iv) [2 marks] If $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is a subset of V, is it a linearly independent set? Justify your answer.

(b) Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ be a basis for $\mathbb{R}^3$.

(i) [5 marks] Show that $T = \{\boldsymbol{u_1} + \boldsymbol{u_2}, \boldsymbol{u_1} - \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a basis for $\mathbb{R}^3$.

(ii) [3 marks] Find the transition matrix from S to T.

(c) [4 marks] Give an example of a family of subspaces $V_1, V_2, \cdots, V_n$ of $\mathbb{R}^n$ such that $\dim V_i = i$ for $i = 1, 2, \cdots, n$ and $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n$. Justify your answer.

Show your working below and on the next three pages.

(Working spaces for Question 1 - Indicate your parts clearly)

(More working spaces for Question 1)

(More working spaces for Question 1)

Question 2 [20 marks]

(a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$.

(i) [2 marks] Find a basis for the row space of $\boldsymbol{A}$.

(ii) [2 marks] Find a basis for the column space of $\boldsymbol{A}$.

(iii) [3 marks] Find a basis for the nullspace of $\boldsymbol{A}$.

(iv) [3 marks] Extend the basis in (iii) to a basis for $\mathbb{R}^5$.

Show working for all the above parts.

(b) [6 marks] Let

$$\boldsymbol{C} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x^2-x-2 & x+1 \end{pmatrix}.$$

Find all the values of x such that:
(i) $\text{rank}(\boldsymbol{C}) = 1$; (ii) $\text{rank}(\boldsymbol{C}) = 2$; (iii) $\text{rank}(\boldsymbol{C}) = 3$.

Justify your answers.

(c) [4 marks] Let $(a\ b\ c)$ be a nonzero vector that belongs to the row space of a 3×3 matrix $\boldsymbol{B}$. Show that the nullspace of $\boldsymbol{B}$ is a subset of the plane $ax + by + cz = 0$.

Show your working below and on the next three pages.

(Working spaces for Question 2 - Indicate your parts clearly)

(More working spaces for Question 2)

(More working spaces for Question 2)

Question 3 [20 marks]

(a) Let $\boldsymbol{r_1} = (2, 1, 3)$, $\boldsymbol{r_2} = (0, 2, 4)$, $\boldsymbol{r_3} = (-2, 1, 1)$ and $V = \text{span}\{\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}\}$.

(i) [3 marks] Show that $S = \{\boldsymbol{r_1}, \boldsymbol{r_3}\}$ forms an orthogonal basis for the vector space V.

(ii) [3 marks] Let $\boldsymbol{u} = (2, 9, 19)$. Find the coordinate vector $(\boldsymbol{u})_S$ with respect to the orthogonal basis S.

(iii) [3 marks] Compute the projection of $\boldsymbol{v} = (2, 4, 2)$ onto V.

Show working for parts (ii) and (iii) above.

(b) Let $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 2 & 1 \\ 3 & 4 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(i) [2 marks] Show that $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

(ii) [4 marks] Find all the least squares solutions to $\boldsymbol{Ax} = \boldsymbol{b}$. (Show working.)

(c) [5 marks] Let $\boldsymbol{B}$ be a $m \times n$ matrix where $m < n$. Prove that for any $\boldsymbol{b} \in \mathbb{R}^m$, the linear system $\boldsymbol{Bx} = \boldsymbol{b}$ has infinitely many least squares solutions .

Show your working below and on the next three pages.

(Working spaces for Question 3 - Indicate your parts clearly)

(More working spaces for Question 3)

(More working spaces for Question 3)

Question 4 [20 marks]

(a) Let $\boldsymbol{A} = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(i) [2 marks] Find all the eigenvalues of $\boldsymbol{A}$.

(ii) [4 marks] Find a basis for each eigenspace of $\boldsymbol{A}$.

(iii) [2 marks] Is $\boldsymbol{A}$ a diagonalizable matrix?

(iv) [3 marks] Find a non-zero diagonal matrix $\boldsymbol{B}$ such that $\boldsymbol{A}+\boldsymbol{B}$ is diagonalizable.

Justify your answers to all the parts above.

(b) [4 marks] Suppose $\boldsymbol{C}$ is a 2×2 matrix such that

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \boldsymbol{C} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Find all eigenvalues of $\boldsymbol{C}^T$ and an eigenvector that corresponds to each eigenvalue. (Show working.)

(c) [5 marks] Prove that if $\boldsymbol{X}$ is a diagonalizable $n \times n$ matrix with only one eigenvalue λ, then $\boldsymbol{X} = \lambda \boldsymbol{I_n}$.

Show your working below and on the next three pages.

(Working spaces for Question 4 - Indicate your parts clearly)

(More working spaces for Question 4)

(More working spaces for Question 4)

Question 5 [20 marks]

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation given by the formula

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+z \\ y-z \end{pmatrix}$$

(i) [2 marks] Write down the standard matrix $\boldsymbol{A}$ of T.

(ii) [3 marks] Write down an explicit set notation for $\ker(T)$.

(iii) [3 marks] Write down the rank and nullity of T.

(iv) [2 marks] Write down the rank and nullity of the linear transformation $T' : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T'(\boldsymbol{v}) = \boldsymbol{A}^T\boldsymbol{v}$.

Show working for all the parts above.

(v) [2 marks] Is the range $R(T)$ equal to $\mathbb{R}^2$?

(vi) [3 marks] Is it possible to find a linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^3$ such that $S \circ T$ is the identity transformation?

Justify your answers for parts (v) and (vi).

(b) [5 marks] Let $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ be an orthonormal basis for $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation such that $\{F(\boldsymbol{u_1}), F(\boldsymbol{u_2}), \ldots, F(\boldsymbol{u_n})\}$ is also an orthonormal basis. Show that the standard matrix of F is an orthogonal matrix.

Show your working below and on the next three pages.

(Working spaces for Question 5 - Indicate your parts clearly)

(More working spaces for Question 5)

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