NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2008-2009

QF4102 — Financial Modelling

November/December 2008 **Time allowed: 2.5 hours.**

INSTRUCTIONS TO CANDIDATES

1. Including this page, the examination paper comprises **FOUR(4)** printed pages.
2. This exam has 2 sections:
   - Section A: Four compulsory questions. Answer **ALL** questions.
   - Section B: Two optional questions. Choose **ONE** question only.
3. Each question carries 20 marks.

Section A [80%]: (Compulsory questions) Answer all questions.

Question 1 [20 marks]

Consider a European-style arithmetic Asian option with payoff

$$\left(\frac{1}{T}\int_0^T S_\tau d\tau - X\right)^+,$$

where S_t is the underlying asset price, T the maturity and X the strike price. Assume that S_t follows a geometric Brownian motion. Denote the option value by $V = V(S_t, I_t, t)$, where

$$I_t = \int_0^t S_\tau d\tau$$

is a path-dependent variable.

(i) [5 marks] Write the partial differential equation model for $V(S, I, t)$.

(ii) [5 marks] Is any boundary condition needed at $I = 0$? Why or why not? How do we discretize the term $\frac{\partial V}{\partial I}$ in the model by the finite difference method?

(iii) [10 marks] Give the binomial tree method and the forward shooting grid method with linear interpolation for the option pricing.

Question 2 [20 marks]

Consider the following pricing model of a European-style option on two assets S_1 and S_2:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + rS_1\frac{\partial V}{\partial S_1} + rS_2\frac{\partial V}{\partial S_2} - rV = 0$$

in $S_1 > 0$, $S_2 > 0$, $t \in [0, T)$, with the terminal condition

$$V(S_1, S_2, T) = (X - \max(S_1, S_2))^+ .$$

Here, r, σ_1, σ_2, ρ and X are all positive constants, and $\rho \in (0, 1)$.

(i) [10 marks] Give the fully implicit finite difference scheme (the boundary conditions should be prescribed).

(ii) [10 marks] Write the corresponding American-style option pricing model and present a penalty method with the Crank-Nicolson scheme.

Question 3 [20 marks]

Consider a European vanilla call option written on a stock whose price follows a geometric Brownian motion. Suppose that short selling of the stock is not permitted. Assume the exponential utility, i.e., the utility function $U(x) = -e^{-\alpha x}$, $\alpha > 0$.

(i) [5 marks] Write the utility indifference ask price (from the point of view of the option's writer) and bid price (from the point of view of the option's buyer) of the option, respectively.

(ii) [5 marks] Give a dynamic programming algorithm for computing the bid price.

(iii) [10 marks] Study the relation between the ask price and the Black-Scholes price: are they equal or not? If not, which is bigger? Give the reason for your answer. Here the Black-Scholes price is the option price computed from the Black-Scholes model.

Question 4 [20 marks]

Consider a regime switching market in which there are two regimes: "Bull market" (regime 1) and "Bear market" (regime 2). The risk neutral process of the stock price follows

$$\frac{dS_t}{S_t} = r_i dt + \sigma_i dB_t,\ \ i \in \{1, 2\},$$

where B_t is a standard Brownian motion, r_i and σ_i respectively represent the risk free rate and the stock volatility in regime $i \in \{1, 2\}$. It is assumed that regime i switches into regime j at the first jump time of a Poisson process with intensity λ_i, for $i, j \in \{1, 2\}$, $i \neq j$. There is no correlation between the Brownian motion and the Poisson process.

(i) [15 marks] Write the algorithm for the Monte Carlo method of pricing a European vanilla call option written on the stock in the market.

(ii) [5 marks] Describe how to find the delta of the option in your algorithm.

Section B [20%]: (Optional questions) Answer one question only.

Question 5 [20 marks]

(a) [15 marks] Describe the Monte-Carlo method for pricing the American put option.

(b) [5 marks] Explain the technique of antithetic variance reduction.

Question 6 [20 marks]

Use a two-step trinomial tree to illustrate the implied tree method.

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