NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2008-2009

MA5205 Graduate Analysis I

November 27, 2008 — Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper comprises **Three (3)** printed pages.
2. This paper consists of **FIVE (5)** questions. Answer **ALL** of them.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Answer **all** *the questions in this paper*

Question 1 [25 marks]

(i) Show that if f is continuous over $(a, b]$ except at $x = a$ and if f is Lebesgue integrable over $[a, b]$, then f is improper integrable, i.e, the limit $\lim_{t\to a^+} \int_t^b f(x)dx$ exists and is finite.

(ii) Show that $\frac{1}{x}$ is not Lebesgue integrable over $[0, 1]$.

(iii) Show that if $f(x)$ is Lebesgue integrable over $[0, a]$, then for any $k > 0$, $f(kx)$ is integrable over $[0, a/k]$.

(iv) Show that $f(x) = \frac{1}{x}\cos(\frac{2}{x})$ if $x \in (0, 1]$ and $f(0) = 0$ is not Lebesgue integrable, but its improper integral exists. (**Hint:** If this is Lebesgue integrable, by (iii), $g(x) := \frac{1}{x}\cos(\frac{1}{x})$ would also be integrable, so is $\frac{1}{x}\cos^2(\frac{1}{x})$ by observation that it is less than $|g(x)|$. Try to compare with (ii).)

Question 2 [20 marks]

Define $f(x) = x^2$ if x is a rational number in the interval $[0, 1]$ and $f(x) = x^3$ if x is an irrational number in the same interval.

(i) Show that for all $t > 0$, $\ln(1 + e^t) < t + \ln 2$.

(ii) Use (i) or otherwise to show that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \ln(1 + e^{nf(x)})dx = \frac{1}{4}.$$

Question 3 [20 marks]

Show that if $f \in L^p(R^n)$ and $K \in L^{p'}(R^n)$ with $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, then the convolution $f * K$ is bounded and continuous in R^n.

Question 4 [15 marks]

Suppose $f_k \to f$ in $L^p(E)$, $1 \le p < \infty$, for a measurable set E and $g_k \to g$ point-wise in E, $\|g_k\|_{L^\infty} \le M$ for all k. Show that $f_k g_k \to fg$ in $L^p(E)$.

Question 5 [20 marks]

Show that if $f \in L^1(R^n)$, then the maximum function $f^* \in L^q(E)$ for any $0 < q < 1$ and any measurable set E with finite measure. (**Hint:** Let $\omega(\alpha) = |\{x \in R^n : f^*(x) > \alpha\} \cap E|$. Then use the identity $\int_E (f^*)^q = q \int_0^\infty \alpha^{q-1} \omega(\alpha) d\alpha$.)

END OF PAPER