NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2008-2009

MA5203 Graduate Algebra I

November/December 2008 – Time allowed : 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **THREE (3)** printed pages.
2. Answer not more than **FOUR (4)** questions.
3. Maximum marks will be allocated as follows:

Your best 3 answers	:	28% each	84%
Next best answer	:	16%	16%

			100%

4. Candidates may use their own notes, but not published material.
5. Results proved in lectures or tutorial assignments that you use should be stated clearly but need not be proved.
6. The symbol R always refers to a ring (with 1), assumed arbitrary, unless otherwise stated.

Question 1. Show that any simple group of order (12)p with p prime has order 60. [*Hint.* Consider separately each case where $p \leq 11$, and $p \geq 13$.]

Question 2. Let R be a ring with invariant basis number.

(i) Write down (without proof) a basis for $\mathrm{End}(R^m_R)$.

Now suppose that $A \in M_m(R)$ and $B \in M_n(R)$.

(ii) Show that there is a polynomial $f(t) \in R[t]$ of degree $\leq m^2$ that vanishes on A; that is, $f(A) = 0 \in \mathrm{End}(R^m_R)$.

(iii) Show that if $f(t)$ vanishes on A and $g(t)$ vanishes on B, where $f(t), g(t) \in R[t]$ are coprime, then $f(B)$ is invertible.

Now let M be the right $R[t]$-module R^m on which t acts as A, and let N be the right $R[t]$-module R^n on which t acts as B.

(iv) Show that $\mathrm{Hom}(M_{R[t]}, N_{R[t]}) = 0$ if there are coprime polynomials $f(t), g(t) \in R[t]$ such that $f(t)$ vanishes on A and $g(t)$ vanishes on B.

Question 3. **(a)** Let $f : R \to S$ be a ring homomorphism.

(i) Let M be a right S-module. Show that if M is Artinian as a right R-module via f (also written $f^{\#}M$), then M is Artinian as a right S-module.

(ii) Show that if f is surjective and R is a right Artinian ring, then so is S. [*Hint.* Consider the exact sequence of right R-modules

$$0 \to \mathrm{Ker} f \longrightarrow R \longrightarrow S \to 0$$

in $\textsc{Mod}_R$.]

(b) Let R be a right Artinian ring.

(i) Show that if $a \in R$, then there is an integer k and some $b \in R$ with $a^k = a^{k+1}b$.

(ii) Deduce that if R is a (not necessarily commutative) domain, then it is a division ring.

(c) Show that in a right Artinian ring every prime twosided ideal is maximal.

Question 4. Let Pair_R be the category whose objects are pairs (M, N) where M is a right R-module with submodule N, and whose morphisms $(M_1, N_1) \to (M_2, N_2)$ are module homomorphisms $\phi : M_1 \to M_2$ such that $\phi(N_1) \subseteq N_2$.

Let SES_R be the category whose objects are short exact sequences of right R-modules and morphisms are commuting diagrams

$$\begin{array}{ccccccccc} 0 & \to & M_1' & \to & M_1 & \to & M_1'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_2' & \to & M_2 & \to & M_2'' & \to & 0 \end{array}$$

(a) By considering the short exact sequence

$$0 \to \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \to 0$$

or otherwise, show that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ does not define a functor from $\text{SES}_{\mathbb{Z}}$ to itself.

(b) Prove that the forgetful functor $\Upsilon : \text{SES}_R \to \text{Pair}_R$ sending

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

to $(M, \alpha(M'))$ is an equivalence of categories.

(c) Define a functor $S : \text{Pair}_R \to \text{SES}_R$ by sending (M, N) to

$$0 \to N \longrightarrow N \oplus M/N \longrightarrow M/N \to 0.$$

For consideration of whether S is (either left or right) adjoint to Υ, what are the morphism groups that arise in respect of the object of $\text{SES}_{\mathbb{Z}}$ given in (a) above? [You need not give proofs.]

Is S (either left or right) adjoint to Υ?

Question 5. (a) Let $f : R \to S$ be a ring homomorphism, and M a finitely generated flat right R-module. Show that $f_\# M = M \otimes_R S$ is a finitely generated flat right S-module.

(b) Suppose that R is a commutative ring. Prove that a right R-module L is a flat R-module if and only if for every prime ideal $\mathfrak{p}$ of R the localization $L_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module.

END OF PAPER