NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2008-2009

MA4262 Measure and Integration

Nov/Dec 2008 — Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in this paper. Marks for each question are indicated at the beginning of the question.

Notation. Throughout this paper the symbols $\mathbb{R}$, m^* and m denote the real line, the Lebesgue outer measure and the Lebesgue measure, respectively.

Answer **all** *the questions. Marks for each question are indicated at the beginning of the question.*

Question 1 [15 marks]

(a) Let Ω be the collection of all subsets A of $\mathbb{R}$ with

$$m^*(A) = \sup\{m(F) \,:\, F \subseteq A \text{ and } F \text{ is a bounded and closed subset of } \mathbb{R}\}.$$

Prove or disprove that every set in Ω is Lebesgue measurable.

(b) Let (f_j) be a sequence of real-valued functions on $\mathbb{R}$ and let $E_{jk} := \{x \in \mathbb{R} : f_j(x) \geq 1/k\}$. Prove that

$$\{x \in \mathbb{R} : \limsup_{j\to\infty} f_j(x) > 0\} = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{j=N}^{\infty} E_{jk}.$$

(c) Let f be a real-valued Lebesgue measurable function on $\mathbb{R}$. If A is a G_δ set in $\mathbb{R}$, prove or disprove that the set $\{x \in \mathbb{R} \,:\, f(x) \in A\}$ is Lebesgue measurable.

Question 2 [15 marks]

(a) Let E be a subset of $\mathbb{R}$. Prove that E is Lebesgue measurable if and only if for every $\epsilon > 0$, there exist open sets G_1 and G_2 of $\mathbb{R}$ such that $E \subseteq G_1$, $\mathbb{R}\backslash E \subseteq G_2$ and $m(G_1 \cap G_2) < \epsilon$.

(b) Let $f(x, y)$ be a real-valued function on $\mathbb{R}^2$ such that for each fixed y in $\mathbb{R}$, the function $f(x, y)$ is measurable (as a function of x) in $\mathbb{R}$; and for each fixed x in $\mathbb{R}$, the function $f(x, y)$ is continuous (as a function of y) in $\mathbb{R}$. Define the function F on $\mathbb{R}$ by

$$F(x) := \max_{0\leq y\leq 1} f(x, y).$$

Prove that F is Lebesgue measurable on $\mathbb{R}$.

Question 3 [15 marks]

(a) Let f be a real-valued Lebesgue measurable function on $\mathbb{R}$ and let ϕ be an increasing function on $\mathbb{R}$. Prove or disprove that the composite function $\phi \circ f$ is Lebesgue measurable on $\mathbb{R}$.

(b) Let f be a real-valued Lebesgue measurable function on a Lebesgue measurable set E. If $m(E) < \infty$, prove that for any $\epsilon > 0$, there exists a bounded Lebesgue measurable function g on E such that

$$m\big(\{x \in E : |f(x) - g(x)| > 0\}\big) < \epsilon.$$

(c) Let $f : \mathbb{R} \to [0, \infty]$ be a Lebesgue measurable function. Prove that there exists an increasing sequence (s_n) of simple (Lebesgue measurable) functions such that $\lim_{n\to\infty} s_n = f$ and for each n, the set $\{x \in \mathbb{R} : s_n(x) > 0\}$ is of finite Lebesgue measure.

Question 4 [15 marks]

(a) Let f be a real-valued function on $[a, b]$ and let $E = \{x \in [a, b] : f'(x) \text{ exists}\}$ where $f'(x)$ is the derivative of f at x. If $|f'(x)| \leq 1$ for all $x \in E$, prove that

$$m^*(f(E)) \leq m^*(E).$$

(b) Let f be a Lebesgue integrable function on $\mathbb{R}$ and let $E_n = \{x \in \mathbb{R} : |f(x)| \geq n\}$ for $n = 1, 2, 3, \cdots$. Evaluate $\lim_{n\to\infty} n\, m(E_n)$.

(c) Let f be a Lebesgue integrable function on $\mathbb{R}$. If

$$\int_R f(x)\,\phi(x)\,dx = 0$$

for all bounded Lebesgue measurable functions ϕ on $\mathbb{R}$, prove that $f = 0$ almost everywhere on $\mathbb{R}$.

Question 5 [20 marks]

(a) Let $f \in L_p(\mathbb{R})$ and $g \in L_q(\mathbb{R})$ where $1 < p, q < \infty$ and $1/p + 1/q = 1$. Let F be the function defined on $\mathbb{R}$ by

$$F(t) = \int_{\mathbb{R}} f(x+t)g(x)\,dx.$$

Prove that F is continuous on $\mathbb{R}$.

(b) Evaluate

$$\lim_{n\to\infty} \int_0^{2\pi} |\sin x - \cos x|^{1/n}\,dx.$$

(c) Let $f \in L_{p_1}(\mathbb{R})$ and $g \in L_{p_2}(\mathbb{R})$ where $1 < p_1, p_2 < \infty$. Prove that there exists a positive real number p such that the pointwise product fg is in $L_p(\mathbb{R})$.

Question 6 [20 marks]

(a) Let $n_1, n_2, n_3, \cdots$ be an increasing sequence of positive integers. Let $E = \{x \in [0, 2\pi] : \lim_{k\to\infty} \sin n_k x \text{ exists}\}$. Prove that E is Lebesgue measurable and evaluate $m(E)$.

(b) Prove that a real-valued function f is of bounded variation on $[a, b]$ if and only if there exists an increasing function ϕ on $[a, b]$ such that

$$f(y) - f(x) \leq \phi(y) - \phi(x)$$

for all $a \leq x < y \leq b$.

(c) Let f be a function of bounded variation on $[0, 1]$. Define the function F on $[0, 1]$ by

$$F(x) = \begin{cases} \frac{1}{x}\int_0^x f(t)\,dt & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove or disprove that F is of bounded variation on $[0, 1]$.

END OF PAPER