NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2008-2009

MA4247 Complex Analysis II

December 2008 — Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of a total of **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.
3. Candidates may use non-graphing, non-programmable calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) Determine the total number of zeroes of the polynomial

$$p(z) = i(z^7 - 2z^5) + z^3 - 1,$$

that lie in the upper half plane. Justify your answer.

(b) Let $g(z)$ be given by the following expression

$$g(z) = \frac{(z^2 - 2)(z - 3)e^z}{(z + 1)^2}.$$

Evaluate the following integral about the positively oriented contour $\{|z| = 2\}$,

$$\int_{|z|=2} \frac{z^2 g'(z)}{g(z)} dz.$$

Question 2 [20 marks]

(a) Let $f(z)$ and $g(z)$ be analytic functions defined on a domain D. Prove that if

$$f(z)g(z) = 0, \qquad \text{for all } z \in D,$$

then either $f(z) = 0$ for all $z \in D$ or $g(z) = 0$ for all $z \in D$.

(b) Let $f(z)$ and $g(z)$ be *entire* functions. Is it true that if $f(z)g(z) = k$ for all $z \in \mathbb{C}$ and some non zero complex constant k, then either $f(z)$ or $g(z)$ is constant? Justify your answer.

(c) Let $f(z)$ and $g(z)$ be *entire* functions. Is it true that if

$$f\big(g(z)\big) = 0, \qquad \text{for all } z \in \mathbb{C},$$

then either $g(z)$ is constant or $f(z) = 0$ for all $z \in \mathbb{C}$? Justify your answer.

(d) Let $\{a_n\}$ and $\{b_n\}$ be two complex sequences such that

$$F(z) = \sum_{n=1}^{\infty} a_n z^n \qquad \text{and} \qquad G(z) = \sum_{n=1}^{\infty} b_n z^n$$

are two convergent power series in the unit disk $B(0, 1)$. Suppose the equation

$$F(z) = G(z)$$

has infinitely many solutions in $B(0, 1)$. Is it true that $a_n = b_n$ for all n? Justify your answer.

Question 3 [20 marks]

(a) Find a linear fractional transformation $T(z)$ that maps the unit disk $B(0,1)$ conformally onto $B(1,2)$, such that $T(0) = 1+i$ and $T(1) = 1-2i$. Is this transformation unique?

(b) Let W be the domain given by

$$W = \Big\{ z : |z| < 1 \text{ and } |z-(1+i)| > 1 \Big\}.$$

Find a conformal mapping from W onto $B(0,1)$. You may leave your answer as a composition of mappings.

(c) Let U be the half-plane given by the following equation

$$U = \Big\{ z : |z-(i-1)| < |z| \Big\}.$$

and S be the linear fractional transformation given by

$$S(z) = \frac{2z}{z-i}.$$

Find and sketch the image of U under the transformation $S(z)$.

Question 4 [20 marks]

(a) Prove that the following infinite products converge and evaluate the limit.

(i) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{2}{n(n+3)}\right);$

(ii) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{4}{(2n+1)^2}\right).$

(b) Using Weierstrass' theorem or otherwise, construct an entire function $f(z)$ such that

$$f(z) = 0 \quad \text{if and only if} \quad z = n,$$

for all non negative integers $n = 0, 1, 2 \ldots.$

(c) Show that

$$\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2 = \frac{\pi}{\cosh \pi y},$$

for all $y \in \mathbb{R}$.

Question 5 [20 marks]

(a) Show that the function $h(z) : \mathbb{C} \to B(0,1)$ defined by

$$h(z) = \frac{z}{1+|z|},$$

is a bijection, i.e. it is one-one and onto. Explain why this does not contradict the Riemann mapping theorem.

(b) Let $\phi(z)$ be a real valued continuous function on a domain D. We say $\phi(z)$ is *superharmonic* if for every closed ball $\overline{B(z_0, r)} \subseteq D$,

$$\phi(z_0) \geq \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + re^{i\theta}) d\theta.$$

Likewise, $\phi(z)$ is *subharmonic* if for every closed ball $\overline{B(z_0, r)} \subseteq D$,

$$\phi(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z_0 + re^{i\theta}) d\theta.$$

(i) Prove the following minimum principle for superharmonic functions:
Let D be a domain and $\phi(z)$ be a superharmonic function defined on D. If there exists $z_0 \in D$, such that

$$\phi(z_0) \leq \phi(z) \quad \text{for all } z \in D,$$

then $\phi(z)$ must be a constant function.

(ii) Let $f(z)$ be defined by

$$f(z) = |z|^2, \quad \text{for } z \in B(0,5).$$

Is $f(z)$ harmonic, subharmonic or superharmonic?

END OF PAPER