NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2008-2009

MA4230 Matrix Computation

December 2008 — Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and comprises **THREE (3)** printed pages (including this page).
2. Answer **ALL** questions. The marks for the questions are not necessarily the same; the mark for each question is indicated at the beginning of the question.
3. Candidates may use one non-programmable calculator. However, they should lay out systematically the various steps in the calculations.
4. For computational questions, candidates may give their answers in exact value **or** in 4 significant figures.

Question 1 [16 marks] Reduce the following matrix to an orthogonally similar upper Hessenberg matrix:

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & -4 \end{bmatrix}.$$

Question 2 [16 marks] Let H be an n-by-n (possibly non-symmetric) upper Hessenberg matrix. Consider applying one step of the QR algorithm with a shift μ to H. That is, factorize $H - \mu I$ into QR, then compute $H^{(1)}$ as $RQ + \mu I$. Show that $H^{(1)}$ is upper Hessenberg.

Question 3 [20 marks] Let A be a real symmetric matrix. Let $r(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}/\|\mathbf{x}\|_2^2$ be the Rayleigh quotient at $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Let $\{\mathbf{v}^{(k)}\}$ and $\{\lambda^{(k)}\}$ be the sequences of approximate eigenvectors and eigenvalues generated by the Rayleigh Quotient Iteration. Let $\mathbf{r}^{(k)} := A\mathbf{v}^{(k)} - \lambda^{(k)}\mathbf{v}^{(k)}$ be the residual vector after the k-th iteration.

(a) Prove that $r(\mathbf{x})$ is the unique minimizer of the function $f(\alpha) := \|A\mathbf{x} - \alpha\mathbf{x}\|_2$, where $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. (6 marks)

(b) Prove that $\|A\mathbf{v}^{(k+1)} - \lambda^{(k)}\mathbf{v}^{(k+1)}\|_2 = |\langle A\mathbf{v}^{(k+1)} - \lambda^{(k)}\mathbf{v}^{(k+1)}, \mathbf{v}^{(k)}\rangle|$. (6 marks)

(c) Prove that the residual norm is monotonically decreasing, i.e. $\|\mathbf{r}^{(k+1)}\|_2 \le \|\mathbf{r}^{(k)}\|_2$. (8 marks)

Question 4 [24 marks] Let A be an n-by-n non-singular matrix. Let $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ be the function $f_1(\mathbf{x}) = A\mathbf{x}$. Let $f_2 : \mathbb{R}^n \to \mathbb{R}^n$ be the function $f_2(\mathbf{x}) = A^{-1}\mathbf{x}$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the function $f(\mathbf{x}) = f_2(f_1(\mathbf{x}))$. Let $\tilde{f}_i$ be a backward stable algorithm for f_i for $i = 1, 2$. Let $\tilde{f}$ be the algorithm $\tilde{f}(\mathbf{x}) = \tilde{f}_2(\tilde{f}_1(\mathbf{x}))$ where $\mathbf{x} \in \mathbb{R}^n$ is assumed to contain floating point numbers only.

(a) Find the condition number of the problem f with respect to $\|\cdot\|_2$ in both the input and output spaces. (4 marks)

(b) Show that the accuracy (relative error) of the algorithm $\tilde{f}$ is $O(\kappa_2(A)\epsilon_{\text{machine}})$, where $\kappa_2(A)$ is the condition number of A with respect to $\|\cdot\|_2$ and $\epsilon_{\text{machine}}$ is the machine-epsilon. (16 marks)

(c) Is the algorithm $\tilde{f}$ uniformly backward stable for all non-singular matrix A? (4 marks)

Question 5 [24 marks] For each of the problems below, choose a suitable method from the given list of three methods for the problem. Give the advantages of your choice over the other two methods and give the disadvantages of each of the other two methods over your chosen method.

(a) Problem: Find the eigenvector corresponding to the unique largest absolute eigenvalue of a large sparse real symmetric matrix.

Methods: (i) Power Iteration; (ii) Inverse Iteration without Shifts; (iii) QR Algorithm with Wilkinson Shifts. (8 marks)

(b) Problem: Solve the least squares problem $\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2$ whose coefficient matrix A is m-by-n with $m \gg n$ and is upper Hessenberg.

Methods: (i) SVD via Householder QR Algorithm with Wilkinson Shifts; (ii) Householder QR Factorization; (iii) Givens QR Factorization. (8 marks)

(c) Problem: Find the best rank-1 approximation with respect to the Frobenius norm of a 1000-by-1000 real symmetric matrix.

Methods: (i) SVD via Householder QR Algorithm with Wilkinson Shifts; (ii) SVD via Givens QR Algorithm without Shifts; (iii) Power Iteration. (8 marks)

END OF PAPER. GOOD LUCK!